Lecture 4
Probabilistic temporal logics

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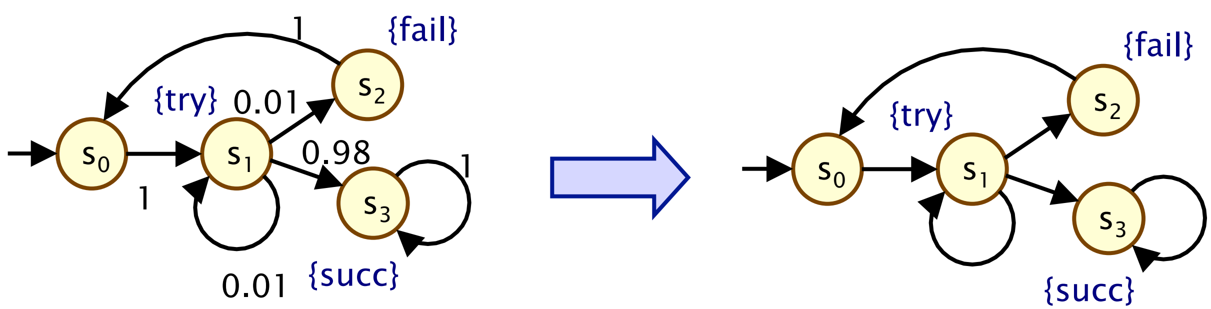
Overview

- Temporal logic
- Non-probabilistic temporal logic
  - CTL
- Probabilistic temporal logic
  - PCTL = CTL + probabilities
- Qualitative vs. quantitative
- Linear-time properties
  - LTL, PCTL*

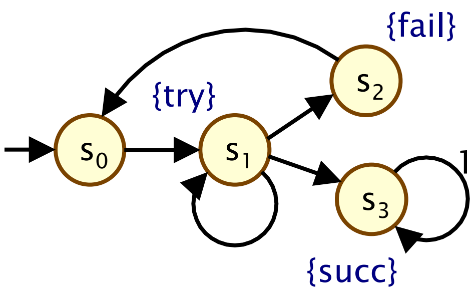
Temporal logic

- Temporal logic
  - formal language for specifying and reasoning about how the behaviour of a system changes over time
  - extends propositional logic with modal/temporal operators
  - one important use: representation of system properties to be checked by a model checker
- Logics used in this course are probabilistic extensions of temporal logics devised for non-probabilistic systems
  - So we revert briefly to (labelled) state-transition diagrams



State–transition systems

- Labelled state–transition system (LTS) (or Kripke structure)
  - is a tuple $(S, s_{init}, \rightarrow, L)$ where:
  - S is a set of states ("state space")
  - $s_{init} \in S$ is the initial state
  - $\rightarrow \subseteq S \times S$ is the transition relation
  - $L : S \rightarrow 2^{AP}$ is function labelling states with atomic propositions (taken from a set AP)



- DTMC $(S, s_{init}, \mathbf{P}, L)$ has underlying LTS $(S, s_{init}, \rightarrow, L)$
  - where $\rightarrow = \{ (s,s') \text{ s.t. } \mathbf{P}(s,s') > 0 \}$

Paths – some notation

- Path $\omega = s_0 s_1 s_2 \ldots$ such that $(s_i, s_{i+1}) \in \rightarrow$ for $i \geq 0$
  - we write $s_i \rightarrow s_{i+1}$ as shorthand for $(s_i, s_{i+1}) \in \rightarrow$
- $\omega(i)$ is the (i+1)th state of ω, i.e. s_i
- $\omega[\ldots i]$ denotes the (finite) prefix ending in the (i+1)th state
  - i.e. $\omega[\ldots i] = s_0 s_{1\ldots} s_i$
- $\omega[i\ldots]$ denotes the suffix starting from the (i+1)th state
  - i.e. $\omega[i\ldots] = s_i s_{i+1} s_{i+2} \ldots$
- As for DTMCs, Path(s) = set of all infinite paths from s

CTL

- CTL – Computation Tree Logic
- Syntax split into state and path formulae
  - specify properties of states/paths, respectively
  - a CTL formula is a state formula

Some of these operators (e.g. A, F, G) are derivable…

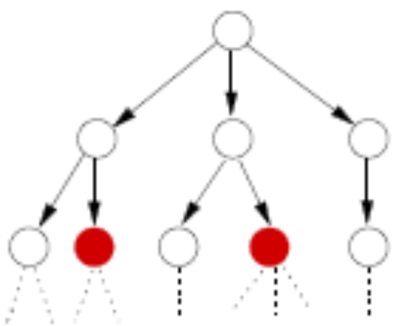
- State formulae:
  - $\phi \ ::= \ \text{true} \mid a \mid \phi \wedge \phi \mid \neg\phi \mid A\ \psi \mid E\ \psi$
  - where $a \in AP$ and ψ is a path formula

- Path formulae
  - $\psi \ ::= \ X\ \phi \mid F\ \phi \mid G\ \phi \mid \phi\ U\ \phi$
  - where ϕ is a state formula

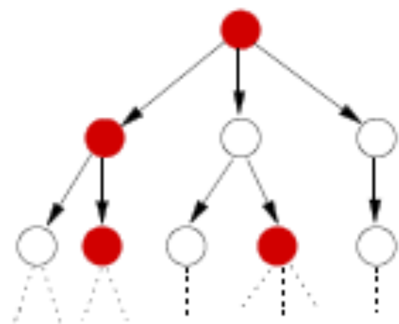
X = "next"
F = "future"
G = "globally"
U = "until"

CTL semantics

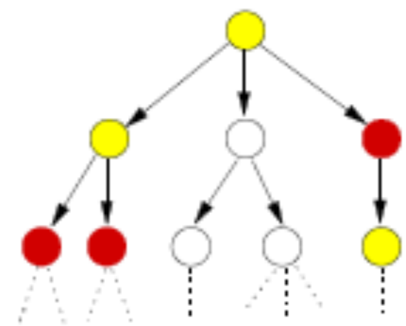
- Intuitive semantics:
  - of quantifiers (A/E) and temporal operators (F/G/U)



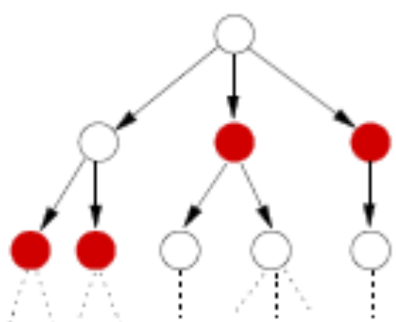
EF red



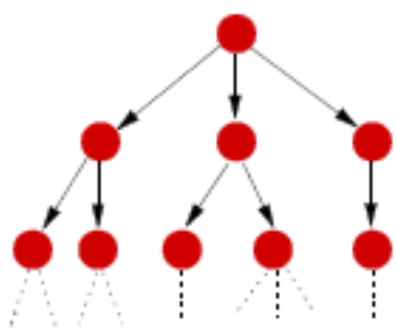
EG red



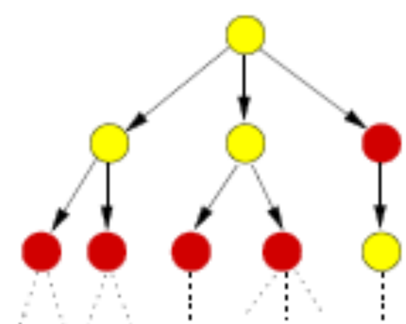
E [yellow U red]



AF red



AG red



A [yellow U red]

CTL semantics

- Semantics of state formulae:
  - $s \models \phi$ denotes "s satisfies ϕ" or "ϕ is true in s"

- For a state s of an LTS $(S,s_{init},\rightarrow,L)$:

  - $s \models true$ always
  - $s \models a \iff a \in L(s)$
  - $s \models \phi_1 \wedge \phi_2 \iff s \models \phi_1$ and $s \models \phi_2$
  - $s \models \neg\phi \iff s \not\models \phi$
  - $s \models A\,\psi \iff \omega \models \psi$ for all $\omega \in Path(s)$
  - $s \models E\,\psi \iff \omega \models \psi$ for some $\omega \in Path(s)$

CTL semantics

- Semantics of path formulae:
  - $\omega \vDash \psi$ denotes "ω satisfies ψ" or "ψ is true along ω"

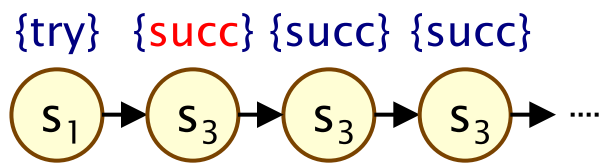
- For a path ω of an LTS $(S, s_{init}, \rightarrow, L)$:

  - $\omega \vDash X\ \phi \quad \Leftrightarrow \quad \omega(1) \vDash \phi$
  - $\omega \vDash F\ \phi \quad \Leftrightarrow \quad \exists k \geq 0 \text{ s.t. } \omega(k) \vDash \phi$
  - $\omega \vDash G\ \phi \quad \Leftrightarrow \quad \forall i \geq 0\ \omega(i) \vDash \phi$
  - $\omega \vDash \phi_1\ U\ \phi_2 \quad \Leftrightarrow \quad \exists k \geq 0 \text{ s.t. } \omega(k) \vDash \phi_2$ and $\forall i < k\ \omega(i) \vDash \phi_1$

CTL examples

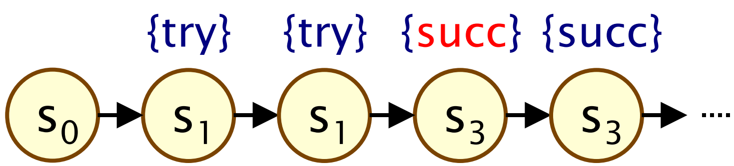
- Some examples of satisfying paths:
  - $\omega_0 \vDash \text{X succ}$

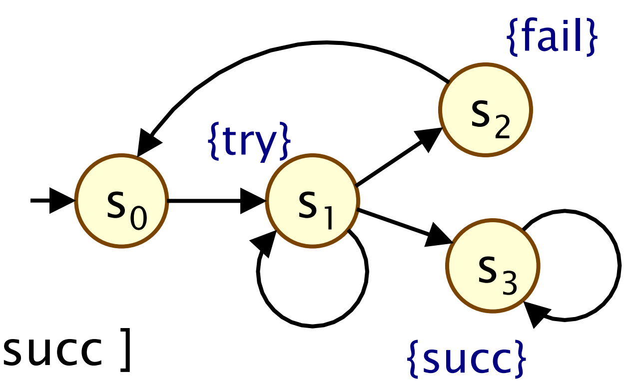
ω_0:



  - $\omega_1 \vDash \neg\text{fail U succ}$

ω_1:





- Example CTL formulas:
  - $s_1 \vDash \text{try} \wedge \neg\text{fail}$
  - $s_1 \vDash \text{E [X succ]}$ and $s_1, s_3 \vDash \text{A [X succ]}$
  - $s_0 \vDash \text{E [}\neg\text{fail U succ]}$ but $s_0 \nvDash \text{A [}\neg\text{fail U succ]}$

CTL examples

- AG $(\neg(crit_1 \wedge crit_2))$
  - mutual exclusion
- AG EF initial
  - for every computation, it is always possible to return to the initial state
- AG (request $\rightarrow$ AF response)
  - every request will eventually be granted
- AG AF $crit_1 \wedge$ AG AF $crit_2$
  - each process has access to the critical section infinitely often

CTL equivalences

- Basic logical equivalences:
  - false $\equiv \neg$true (false)
  - $\phi_1 \vee \phi_2 \equiv \neg(\neg\phi_1 \wedge \neg\phi_2)$ (disjunction)
  - $\phi_1 \rightarrow \phi_2 \equiv \neg\phi_1 \vee \phi_2$ (implication)

- Path quantifiers:
  - $A\ \psi \equiv \neg E(\neg\psi)$
  - $E\ \psi \equiv \neg A(\neg\psi)$

- Temporal operators:
  - $F\ \phi \equiv \text{true}\ U\ \phi$
  - $G\ \phi \equiv \neg F(\neg\phi)$

For example:

$AG\ \phi \equiv \neg EF(\neg\ \phi)$

CTL – Alternative notation

- Some commonly used notation…

- Temporal operators:
  - $F\ \phi \equiv \Diamond\ \phi$ ("diamond")
  - $G\ \phi \equiv \square\ \phi$ ("box")
  - $X\ \phi \equiv \bigcirc\ \phi$
- Path quantifiers:
  - $A\ \psi \equiv \forall\ \psi$
  - $E\ \psi \equiv \exists\ \psi$
- Brackets: none/round/square
  - $AF\ \psi$
  - $A\ (\ \psi_1\ U\ \psi_2\ )$
  - $A\ [\ \psi_1\ U\ \psi_2\ ]$

PCTL

- Temporal logic for describing properties of DTMCs
  - PCTL = Probabilistic Computation Tree Logic [HJ94]
  - essentially the same as the logic pCTL of [ASB+95]
- Extension of (non-probabilistic) temporal logic CTL
  - key addition is probabilistic operator P
  - quantitative extension of CTL's A and E operators
- Example
  - send $\rightarrow$ $P_{\geq 0.95}$ [$F^{\leq 10}$ deliver]
  - "if a message is sent, then the probability of it being delivered within 10 steps is at least 0.95"

PCTL syntax

- PCTL syntax:

  ψ is true with probability $\sim p$

  – ϕ ::= true | a | $\phi \wedge \phi$ | $\neg\phi$ | $P_{\sim p}$ [ψ] (state formulae)

  – ψ ::= X ϕ | ϕ $U^{\leq k}$ ϕ | ϕ U ϕ (path formulae)

  "next" "bounded until" "until"

  – where a is an atomic proposition, $p \in [0,1]$ is a probability bound, $\sim \in \{<,>,\leq,\geq\}$, $k \in \mathbb{N}$

- A PCTL formula is always a state formula
  – path formulae only occur inside the P operator

PCTL semantics for DTMCs

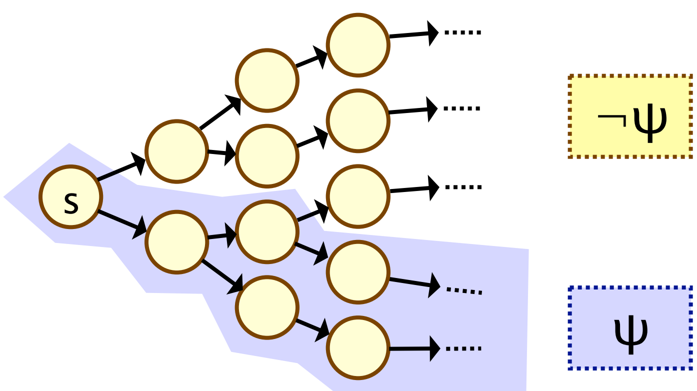
- Semantics for non-probabilistic operators same as for CTL:
  - $s \vDash \phi$ denotes "s satisfies ϕ" or "ϕ is true in s"
  - $\omega \vDash \psi$ denotes "ω satisfies ψ" or "ψ is true along ω"

- For a state s of a DTMC $(S, s_{init}, \mathbf{P}, L)$:
  - $s \vDash \text{true}$ always
  - $s \vDash a \Leftrightarrow a \in L(s)$
  - $s \vDash \phi_1 \wedge \phi_2 \Leftrightarrow s \vDash \phi_1$ and $s \vDash \phi_2$
  - $s \vDash \neg\phi \Leftrightarrow s \nvDash \phi$
- For a path ω of a DTMC $(S, s_{init}, \mathbf{P}, L)$:
  - $\omega \vDash X\,\phi \Leftrightarrow \omega(1) \vDash \phi$
  - $\omega \vDash \phi_1\ U^{\leq k}\ \phi_2 \Leftrightarrow \exists i \leq k$ such that $\omega(i) \vDash \phi_2$ and $\forall j<i,\ \omega(j) \vDash \phi_1$
  - $\omega \vDash \phi_1\ U\ \phi_2 \Leftrightarrow \exists k \geq 0$ s.t. $\omega(k) \vDash \phi_2$ and $\forall i<k\ \omega(i) \vDash \phi_1$

$U^{\leq k}$ not in CTL (but could easily be added)

PCTL semantics for DTMCs

- Semantics of the probabilistic operator P
  - informal definition: $s \vDash P_{\sim p} [\ \psi\ ]$ means that "the probability, from state s, that ψ is true for an outgoing path satisfies ~p"
  - example: $s \vDash P_{<0.25} [\ X\ \mathit{fail}\ ] \Leftrightarrow$ "the probability of atomic proposition fail being true in the next state of outgoing paths from s is less than 0.25"
  - formally: $s \vDash P_{\sim p} [\psi] \ \Leftrightarrow\ \mathrm{Prob}(s, \psi) \sim p$
  - where: $\mathrm{Prob}(s, \psi) = \mathrm{Pr}_s \{\ \omega \in \mathrm{Path}(s) \mid \omega \vDash \psi\ \}$



$\mathrm{Prob}(s, \psi) \sim p$?

PCTL equivalences for DTMCs

- Basic logical equivalences:
  - false $\equiv \neg$true (false)
  - $\phi_1 \vee \phi_2 \equiv \neg(\neg\phi_1 \wedge \neg\phi_2)$ (disjunction)
  - $\phi_1 \rightarrow \phi_2 \equiv \neg\phi_1 \vee \phi_2$ (implication)

- Negation and probabilities
  - e.g. $\neg P_{>p} [\ \phi_1\ U\ \phi_2\ ] \equiv P_{\leq p} [\ \phi_1\ U\ \phi_2\ ]$

Reachability and invariance

- Derived temporal operators, like CTL…

- Probabilistic reachability: $P_{\sim p}$ [F ϕ]
  - the probability of reaching a state satisfying ϕ
  - $F\ \phi \equiv \text{true}\ U\ \phi$
  - "ϕ is eventually true"
  - bounded version: $F^{\leq k}\ \phi \equiv \text{true}\ U^{\leq k}\ \phi$

- Probabilistic invariance: $P_{\sim p}$ [G ϕ]
  - the probability of ϕ always remaining true
  - $G\ \phi \equiv \neg(F\ \neg\phi) \equiv \neg(\text{true}\ U\ \neg\phi)$
  - "ϕ is always true"
  - bounded version: $G^{\leq k}\ \phi \equiv \neg(F^{\leq k}\ \neg\phi)$

strictly speaking, G ϕ cannot be derived from the PCTL syntax in this way since there is no negation of path formulae

Derivation of $P_{\sim p}$ [G ϕ]

- In fact, we can derive $P_{\sim p}$ [G ϕ] directly in PCTL…

PCTL examples

- $P_{<0.05}$ [F err/total>0.1]
  - "with probability at most 0.05, more than 10% of the NAND gate outputs are erroneous?"
- $P_{\geq 0.8}$ [$F^{\leq k}$ reply_count=n]
  - "the probability that the sender has received n acknowledgements within k clock-ticks is at least 0.8"
- $P_{<0.4}$ [$\neg fail_A$ U $fail_B$]
  - "the probability that component B fails before component A is less than 0.4"
- $\neg$oper $\rightarrow$ $P_{\geq 1}$ [F ($P_{>0.99}$ [$G^{\leq 100}$ oper])]
  - "if the system is not operational, it almost surely reaches a state from which it has a greater than 0.99 chance of staying operational for 100 time units"

PCTL and measurability

- All the sets of paths expressed by PCTL are measurable
  - i.e. are elements of the σ-algebra $\Sigma_{Path(s)}$
  - see for example [Var85] (for a stronger result in fact)

- Recall: probability space $(Path(s), \Sigma_{Path(s)}, Pr_s)$
  - $\Sigma_{Path(s)}$ contains cylinder sets $C(\omega)$ for all finite paths ω starting in s and is closed under complementation, countable union

- Next $(X\ \phi)$
  - cylinder sets constructed from paths of length one
- Bounded until $(\phi_1\ U^{\leq k}\ \phi_2)$
  - (finite number of) cylinder sets from paths of length at most k
- Until $(\phi_1\ U\ \phi_2)$
  - countable union of paths satisfying $\phi_1\ U^{\leq k}\ \phi_2$ for all $k \geq 0$

Qualitative vs. quantitative properties

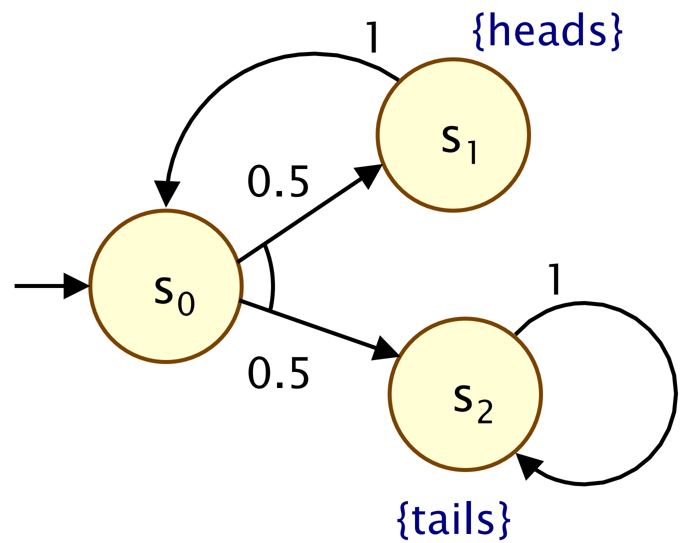
- P operator of PCTL can be seen as a quantitative analogue of the CTL operators A (for all) and E (there exists)

- Qualitative PCTL properties
  - $P_{\sim p}$ [ψ] where p is either 0 or 1
- Quantitative PCTL properties
  - $P_{\sim p}$ [ψ] where p is in the range (0,1)

- $P_{>0}$ [F ϕ] is identical to EF ϕ
  - there exists a finite path to a ϕ-state
- $P_{\geq 1}$ [F ϕ] is (similar to but) weaker than AF ϕ
  - a ϕ-state is reached "almost surely"
  - see next slide…

Example: Qualitative/quantitative

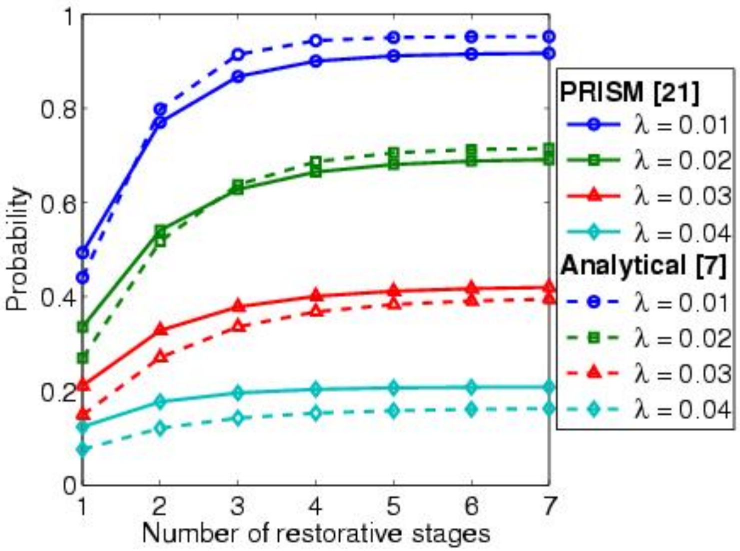
- Toss a coin repeatedly until "tails" is thrown
- Is "tails" always eventually thrown?
  - CTL: AF "tails"
  - Result: false
  - Counterexample: $s_0 s_1 s_0 s_1 s_0 s_1 \ldots$



- Does the probability of eventually throwing "tails" equal one?
  - PCTL: $P_{\geq 1}$ [F "tails"]
  - Result: true
  - Infinite path $s_0 s_1 s_0 s_1 s_0 s_1 \ldots$ has zero probability

Quantitative properties

- Consider a PCTL formula $P_{\sim p}$ [ψ]
  - if the probability is unknown, how to choose the bound p?
- When the outermost operator of a PTCL formula is P
  - PRISM allows formulae of the form $P_{=?}$ [ψ]
  - "what is the probability that path formula ψ is true?"
- Model checking is no harder: compute the values anyway
- Useful to spot patterns, trends
- Example
  - $P_{=?}$ [F err/total>0.1]
  - "what is the probability that 10% of the NAND gate outputs are erroneous?"



Limitations of PCTL

- PCTL, although useful in practice, has limited expressivity
  - essentially: probability of reaching states in X, passing only through states in Y (and within k time-steps)

- More expressive logics can be used, for example:
  - LTL [Pnu77], the non-probabilistic linear-time temporal logic
  - PCTL* [ASB+95,BdA95] which subsumes both PCTL and LTL

- To introduce these logics, we return briefly again to non-probabilistic logics and models…

Branching vs. Linear time

- In CTL, temporal operators always appear inside A or E
  - in LTL, temporal operators can be combined

- LTL but not CTL:
  - F [req $\wedge$ X ack]
  - “eventually a request occurs, followed immediately by an acknowledgement”

- CTL but not LTL:
  - AG EF initial
  - “for every computation, it is always possible to return to the initial state”

LTL

- LTL syntax
  - path formulae only
  - $\psi ::= \ \text{true} \mid a \mid \psi \wedge \psi \mid \neg\psi \mid X\,\psi \mid \psi\,U\,\psi$
  - where $a \in AP$ is an atomic proposition

- LTL semantics (for a path ω)
  - $\omega \vDash \text{true}$ always
  - $\omega \vDash a \Leftrightarrow a \in L(\omega(0))$
  - $\omega \vDash \psi_1 \wedge \psi_2 \Leftrightarrow \omega \vDash \psi_1$ and $\omega \vDash \psi_2$
  - $\omega \vDash \neg\psi \Leftrightarrow \omega \nvDash \psi$
  - $\omega \vDash X\,\psi \Leftrightarrow \omega[1\ldots] \vDash \psi$
  - $\omega \vDash \psi_1\,U\,\psi_2 \Leftrightarrow \exists k \geq 0$ s.t. $\omega[k\ldots] \vDash \psi_2$ and $\forall i<k\ \omega[i\ldots] \vDash \psi_1$

LTL

- LTL semantics
  - implicit universal quantification over paths
  - i.e. for an LTS $M = (S, s_{init}, \rightarrow, L)$ and LTL formula ψ
  - $s \models \psi$ iff $\omega \models \psi$ for all paths $\omega \in Path(s)$
  - $M \models \psi$ iff $s_{init} \models \psi$
- e.g:
  - A F [req $\wedge$ X ack]
  - "it is always the case that, eventually, a request occurs, followed immediately by an acknowledgement"
- Derived operators like CTL, for example:
  - $F\ \psi \equiv \text{true}\ U\ \psi$
  - $G\ \psi \equiv \neg F(\neg\psi)$

LTL + probabilities

- Same idea as PCTL: probabilities of sets of path formulae
  - for a state s of a DTMC and an LTL formula ψ:
  - $\mathrm{Prob}(s, \psi) = \mathrm{Pr}_s \{ \omega \in \mathrm{Path}(s) \mid \omega \vDash \psi \}$
  - all such path sets are measurable (see later)
- Examples (from DTMC lectures)…
- Repeated reachability: "always eventually…"
  - Prob(s, GF send)
  - e.g. "what is the probability that the protocol successfully sends a message infinitely often?"
- Persistence properties: "eventually forever…"
  - Prob(s, FG stable)
  - e.g. "what is the probability of the leader election algorithm reaching, and staying in, a stable state?"

PCTL*

- PCTL* subsumes both (probabilistic) LTL and PCTL

- State formulae:
  - $\phi \ ::= \ \text{true} \mid a \mid \phi \wedge \phi \mid \neg\phi \mid P_{\sim p}\ [\ \psi\ ]$
  - where $a \in AP$ and ψ is a path formula

- Path formulae:
  - $\psi \ ::= \phi \mid \psi \wedge \psi \mid \neg\psi \mid X\ \psi \mid \psi\ U\ \psi$
  - where ϕ is a state formula

- A PCTL* formula is a state formula ϕ
  - e.g. $P_{>0.1}\ [\ GF\ crit_1\ ] \wedge P_{>0.1}\ [\ GF\ crit_2\ ]$

Summing up…

- Temporal logic:
  - formal language for specifying and reasoning about how the behaviour of a system changes over time

CTL LTL	Φ ψ	non-probabilistic (e.g. LTSs)
PCTL LTL + prob. PCTL*	Φ Prob(s, ψ) Φ	probabilistic (e.g. DTMCs)