Lecture 8
Continuous-time Markov chains

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Time in DTMCs

- Time in a DTMC proceeds in discrete steps

- Two possible interpretations:
  - accurate model of (discrete) time units
    - e.g. clock ticks in model of an embedded device
  - time-abstract
    - no information assumed about the time transitions take

- Continuous-time Markov chains (CTMCs)
  - dense model of time
  - transitions can occur at any (real-valued) time instant
  - modelled using exponential distributions

Overview

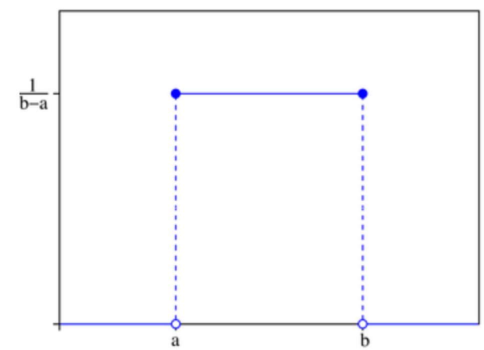
- Exponential distribution and its properties
- Continuous-time Markov chains (CTMCs)
  - definition, examples
  - race condition
  - embedded DTMC
  - generator matrix
- Paths and probabilities
  - probabilistic reachability

Continuous probability distributions

- Defined by:
  - cumulative distribution function

$$F(t) = \Pr(X \le t) = \int_{-\infty}^{t} f(x)\,dx$$

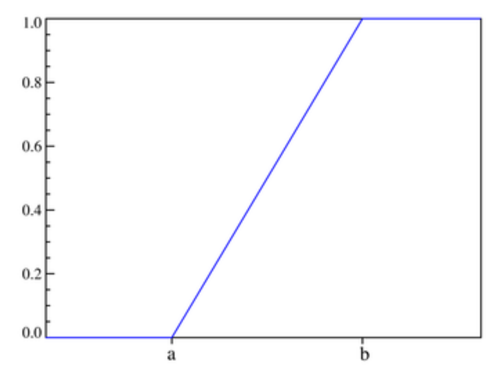
  - where f is the probability density function
  - Pr(X=t) = 0 for all t



- Example: uniform distribution: U(a,b)

$$f(t) = \begin{cases} 1/(b-a) & \text{if } a \le t \le b \\ 0 & \text{otherwise} \end{cases}$$

$$F(t) = \begin{cases} 0 & \text{if } t < a \\ (t-a)/(b-a) & \text{if } a \le t < b \\ 1 & \text{if } t \ge b \end{cases}$$



Exponential distribution

- A continuous random variable X is exponential with parameter λ>0 if the density function is given by:

$$f(t) = \begin{cases} \lambda \cdot e^{-\lambda \cdot t} & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}$$

λ = "rate"

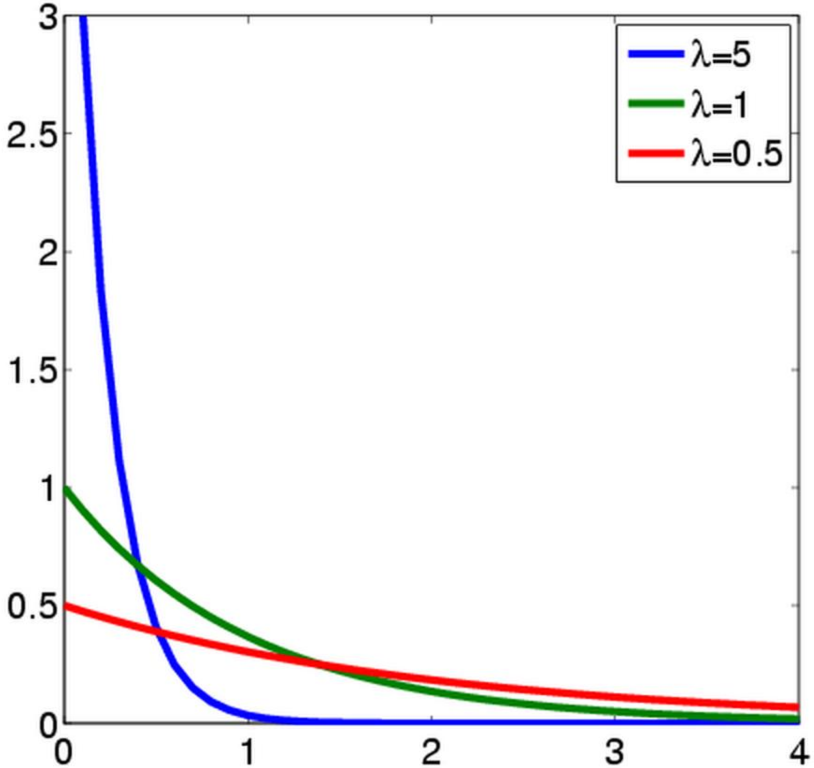
  - we write: X ~ Exponential(λ)
- Cumulative distribution function (for t≥0):

$$F(t) = Pr(X \leq t) = \int_0^t \lambda \cdot e^{-\lambda \cdot x} dx = [-e^{-\lambda \cdot x}]_0^t = 1 - e^{-\lambda \cdot t}$$

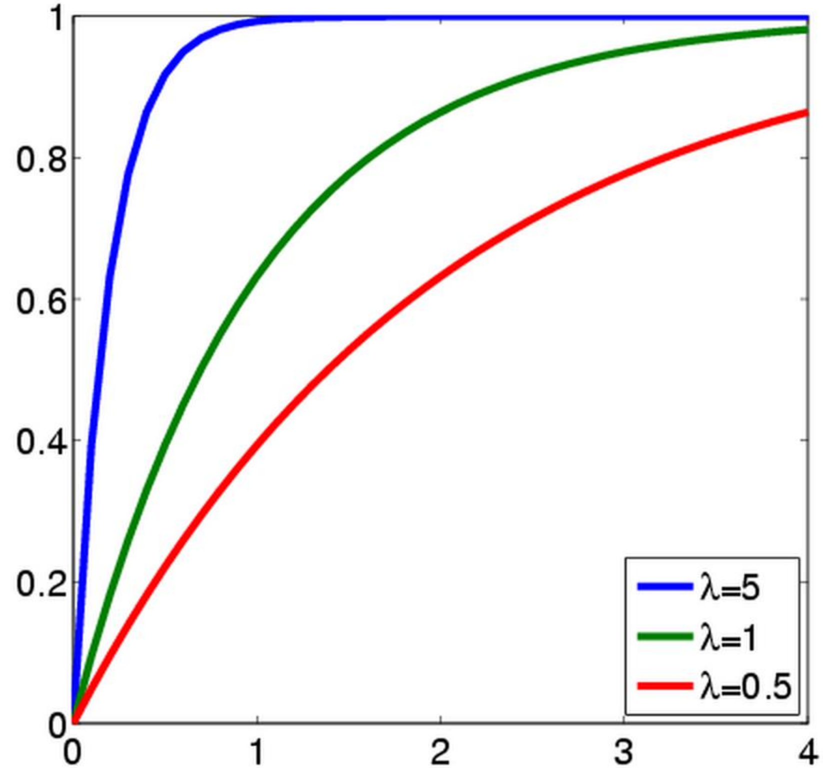
- Other properties:
  - negation: $Pr(X > t) = e^{-\lambda \cdot t}$
  - mean (expectation): $E[X] = \int_0^\infty x \cdot \lambda \cdot e^{-\lambda \cdot x} dx = \frac{1}{\lambda}$
  - variance: $Var(X) = 1/\lambda^2$

Exponential distribution – Examples

Probability density function



Cumulative distribution function



- The more λ increases, the faster the c.d.f. approaches 1

Exponential distribution

- Adequate for modelling many real-life phenomena
  - failures
    - e.g. time before machine component fails
  - inter-arrival times
    - e.g. time before next call arrives to a call centre
  - biological systems
    - e.g. times for reactions between proteins to occur

- Maximal entropy (“uncertainty”) if just the mean is known
  - i.e. best approximation when only mean is known

- Can approximate general distributions arbitrarily closely
  - phase-type distributions

Exponential distribution – Property 1

- The exponential distribution has the memoryless property:
  - $Pr(X>t_1+t_2 \mid X>t_1) = Pr(X>t_2)$

- The exponential distribution is the only continuous distribution which is memoryless
  - discrete-time equivalent is the geometric distribution

Exponential distribution – Property 2

- The minimum of two independent exponential distributions is an exponential distribution (parameter is sum)
  - $X_1 \sim \text{Exponential}(\lambda_1)$, $X_2 \sim \text{Exponential}(\lambda_2)$
  - $Y = \min(X_1, X_2)$
  - $Y \sim \text{Exponential}(\lambda_1 + \lambda_2)$
- Generalises to minimum of n distributions

Exponential distribution – Property 3

- Consider two independent exponential distributions
  - $X_1 \sim$ Exponential(λ_1), $X_2 \sim$ Exponential(λ_2)
  - what is the probability that $X_1 < X_2$?
  - probability that $X_1 < X_2$ is $\lambda_1/(\lambda_1+\lambda_2)$
- Generalises to n distributions

Continuous-time Markov chains

- Continuous-time Markov chains (CTMCs)
  - labelled transition systems augmented with rates
  - discrete states
  - continuous time-steps
  - delays exponentially distributed

- Suited to modelling:
  - reliability models
  - control systems
  - queueing networks
  - biological pathways
  - chemical reactions
  - ...

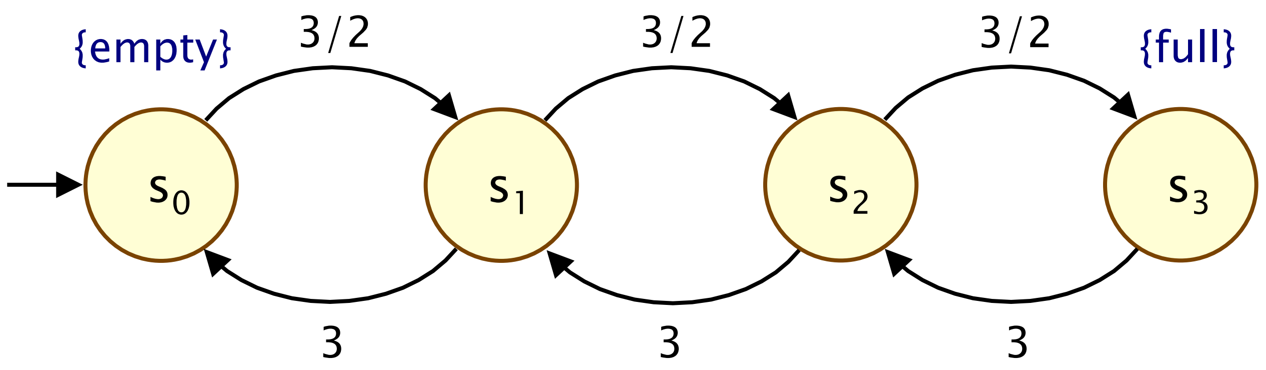
Continuous-time Markov chains

- Formally, a CTMC C is a tuple $(S, s_{init}, \mathbf{R}, L)$ where:
  - S is a finite set of states (“state space”)
  - $s_{init} \in S$ is the initial state
  - $\mathbf{R} : S \times S \rightarrow \mathbb{R}_{\geq 0}$ is the transition rate matrix
  - $L : S \rightarrow 2^{AP}$ is a labelling with atomic propositions

- Transition rate matrix assigns rates to each pair of states
  - used as a parameter to the exponential distribution
  - transition between s and s’ when $\mathbf{R}(s,s')>0$
  - probability triggered before t time units: $1 - e^{-\mathbf{R}(s,s')\cdot t}$

Simple CTMC example

- Modelling a queue of jobs
  - initially the queue is empty
  - jobs arrive with rate 3/2 (i.e. mean inter-arrival time is 2/3)
  - jobs are served with rate 3 (i.e. mean service time is 1/3)
  - maximum size of the queue is 3
  - state space: $S = \{s_i\}_{i=0..3}$ where s_i indicates i jobs in queue



Race conditions

- What happens when there exists multiple s' with **R**(s,s')>0?
  - race condition: first transition triggered determines next state
  - two questions:
  - 1. How long is spent in s before a transition occurs?
  - 2. Which transition is eventually taken?
- 1. Time spent in a state before a transition
  - minimum of exponential distributions
  - exponential with parameter given by summation:

$$E(s) = \sum_{s' \in S} R(s,s')$$

  - probability of leaving a state s within [0,t] is $1 - e^{-E(s) \cdot t}$
  - E(s) is the exit rate of state s
  - s is called absorbing if E(s)=0 (no outgoing transitions)

Race conditions…

- 2. Which transition is taken from state s?
  - the choice is independent of the time at which it occurs
  - e.g. if $X_1 \sim \text{Exponential}(\lambda_1)$, $X_2 \sim \text{Exponential}(\lambda_2)$
  - then the probability that $X_1 < X_2$ is $\lambda_1/(\lambda_1+\lambda_2)$
  - more generally, the probability is given by…
- The embedded DTMC: $emb(C)=(S,s_{init},P^{emb(C)},L)$
  - state space, initial state and labelling as the CTMC
  - for any $s,s' \in S$

$$P^{emb(C)}(s,s') = \begin{cases} R(s,s')/E(s) & \text{if } E(s) > 0 \\ 1 & \text{if } E(s) = 0 \text{ and } s = s' \\ 0 & \text{otherwise} \end{cases}$$

- Probability that next state from s is s' given by $P^{emb(C)}(s,s')$

Two interpretations of a CTMC

- Consider a (non-absorbing) state $s \in S$ with multiple outgoing transitions, i.e. multiple $s' \in S$ with $\mathbf{R}(s,s')>0$

- 1. Race condition
  - each transition triggered after exponentially distributed delay
    - i.e. probability triggered before t time units: $1 - e^{-\mathbf{R}(s,s')\cdot t}$
  - first transition triggered determines the next state

- 2. Separate delay/transition
  - remain in s for delay exponentially distributed with rate E(s)
    - i.e. probability of taking an outgoing transition from s within [0,t] is given by $1-e^{-E(s)\cdot t}$
  - probability that next state is s' is given by $\mathbf{P}^{emb(C)}(s,s')$
    - i.e. $\mathbf{R}(s,s')/E(s) = \mathbf{R}(s,s') / \Sigma_{s' \in S} \mathbf{R}(s,s')$

More on CTMCs…

- Infinitesimal generator matrix Q

$$Q(s,s') = \begin{cases} R(s,s') & s \neq s' \\ -\sum_{s \neq s'} R(s,s') & \text{otherwise} \end{cases}$$

- Alternative definition: a CTMC is:
  - a family of random variables $\{ X(t) \mid t \in \mathbb{R}_{\geq 0} \}$
  - X(t) are observations made at time instant t
  - i.e. X(t) is the state of the system at time instant t
  - which satisfies…

- Memoryless (Markov property)

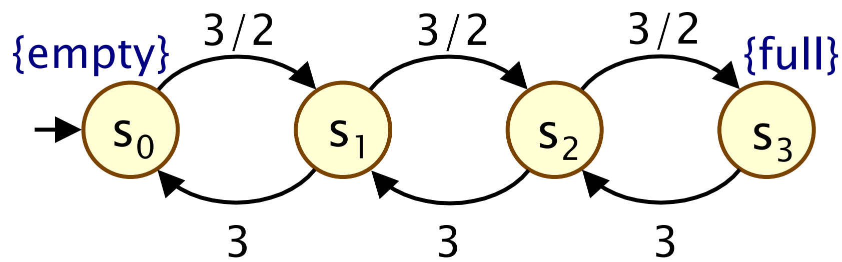
  $\Pr(X(t_k)=s_k \mid X(t_{k-1})=s_{k-1}, \ldots, X(t_0)=s_0) = \Pr(X(t_k)=s_k \mid X(t_{k-1})=s_{k-1})$

Simple CTMC example…

$C = (S, s_{init}, R, L)$

$S = \{s_0, s_1, s_2, s_3\}$

$s_{init} = s_0$



$AP = \{empty, full\}$

$L(s_0)=\{empty\}$, $L(s_1)=L(s_2)=\varnothing$ and $L(s_3)=\{full\}$

$$\mathbf{R} = \begin{bmatrix} 0 & 3/2 & 0 & 0 \\ 3 & 0 & 3/2 & 0 \\ 0 & 3 & 0 & 3/2 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

transition rate matrix

$$\mathbf{P}^{emb(C)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2/3 & 0 & 1/3 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

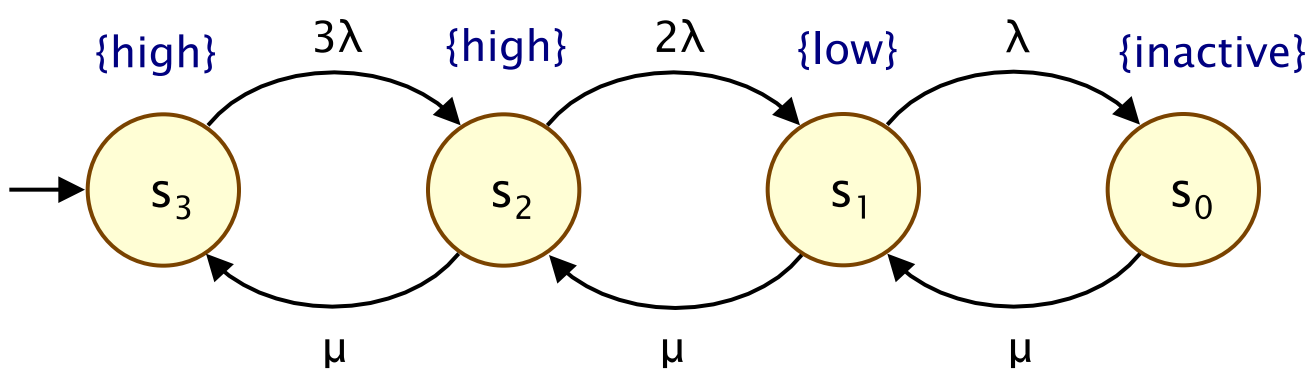
embedded DTMC

$$\mathbf{Q} = \begin{bmatrix} -3/2 & 3/2 & 0 & 0 \\ 3 & -9/2 & 3/2 & 0 \\ 0 & 3 & -9/2 & 3/2 \\ 0 & 0 & 3 & -3 \end{bmatrix}$$

infinitesimal generator matrix

Example 2

- 3 machines, each can fail independently
  - delay modelled as exponential distributions
  - failure rate λ, i.e. mean-time to failure (MTTF) = 1 / λ
- One repair unit
  - repairs a single machine at rate μ (also exponential)
- State space:
  - $S = \{s_i\}_{i=0..3}$ where s_i indicates i machines operational

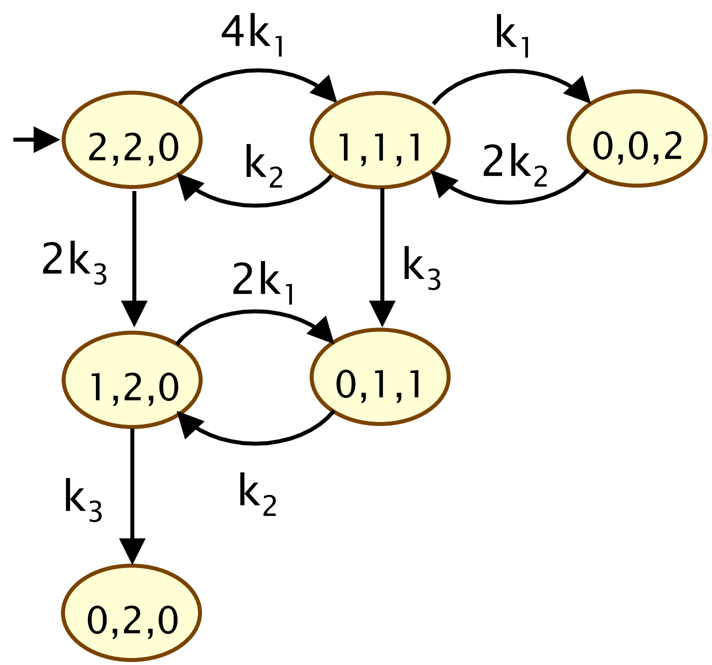


Example 3

- Chemical reaction system: two species A and B
- Two reactions:

$$A + B \underset{k_2}{\overset{k_1}{\rightleftarrows}} AB \qquad A \xrightarrow{k_3}$$

  - reversible reaction under which species A and B bind to form AB (forwards rate = $|A| \cdot |B| \cdot k_1$, backwards rate = $|AB| \cdot k_2$)
  - degradation of A (rate $|A| \cdot k_3$)
  - $|X|$ denotes number of molecules of species X



- CTMC with state space
  - $(|A|,|B|,|AB|)$
  - initially $(2,2,0)$

Paths of a CTMC

- An infinite path ω is a sequence $s_0t_0s_1t_1s_2t_2\ldots$ such that
  - $R(s_i,s_{i+1}) > 0$ and $t_i \in \mathbb{R}_{>0}$ for all $i \in \mathbb{N}$
  - t_i denotes the amount of time spent in s_i
- or a sequence $s_0t_0s_1t_1s_2t_2\ldots t_{k-1}s_k$ such that
  - $R(s_i,s_{i+1}) > 0$ and $t_i \in \mathbb{R}_{>0}$ for all $i<k$
  - s_k is absorbing (i.e. $R(s,s') = 0$ for all $s' \in S$)
  - i.e. remain in state s_k indefinitely

- Path(s) denotes all infinite paths starting in state s
- Further notation:
  - time(ω,j) = amount of time spent in the jth state, i.e. t_j
  - ω@t = state occupied at time t:
  - see e.g. [BHHK03, KNP07a] for precise definitions

Recall: Probability spaces

- A σ-algebra (or σ-field) on Ω is a set Σ of subsets of Ω closed under complementation and countable union, i.e.:
  - if $A \in \Sigma$, the complement $\Omega \setminus A$ is in Σ
  - if $A_i \in \Sigma$ for $i \in \mathbb{N}$, the union $\cup_i A_i$ is in Σ
  - the empty set $\varnothing$ is in Σ
- Elements of Σ are called measurable sets or events
- Theorem: For any set F of subsets of Ω, there exists a unique smallest σ-algebra on Ω containing F
- Probability space (Ω, Σ, Pr)
  - Ω is the sample space
  - Σ is the set of events: σ-algebra on Ω
  - $Pr : \Sigma \to [0,1]$ is the probability measure:

    $Pr(\Omega) = 1$ and $Pr(\cup_i A_i) = \Sigma_i Pr(A_i)$ for countable disjoint A_i

Probability space

- Sample space: Path(s) (set of all paths from a state s)
- Events: sets of infinite paths
- Basic events: cylinders
  - cylinders = sets of paths with common finite prefix
  - include time intervals in cylinders

- Finite prefix is a sequence $s_0,I_0,s_1,I_1,\ldots,I_{n-1},s_n$
  - $s_0,s_1,s_2,\ldots,s_n$ sequence of states where $\mathbf{R}(s_i,s_{i+1})>0$ for $i<n$
  - $I_0,I_1,I_2,\ldots,I_{n-1}$ sequence of of non-empty intervals of $\mathbb{R}_{\geq 0}$

- Cylinder $Cyl(s_0,I_0,s_1,I_1,\ldots,I_{n-1},s_n)$ is the set of infinite paths:
  - $\omega(i)=s_i$ for all $i \leq n$ and $time(\omega,i) \in I_i$ for all $i < n$

Probability space

- Define probability measure over cylinders inductively
- $Pr_s(Cyl(s))=1$
- $Pr_s(Cyl(s,I,s_1,I_1,\ldots,I_{n-1},s_n,I',s'))$ equals:

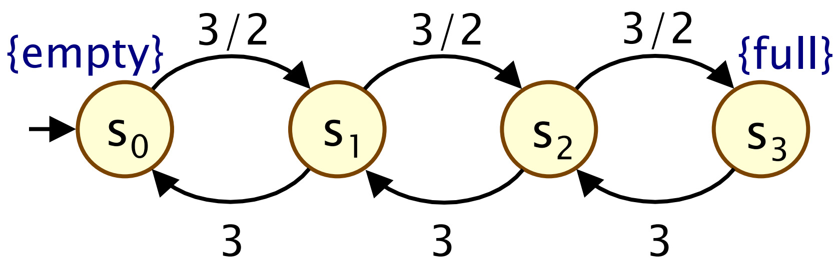
$$Pr_s(Cyl(s,I,s_1,I_1,\ldots,I_{n-1},s_n)) \cdot P^{emb(C)}(s_n,s') \cdot \left(e^{-E(s_n)\cdot \inf I'} - e^{-E(s_n)\cdot \sup I'}\right)$$

probability of transition from s_n to s' (defined using embedded DTMC)

probability time spent in state s_n is within the interval I'

Probability space – Example

- Probability of leaving the initial state s_0 and moving to state s_1 within the first 2 time units of operation



- Cylinder $Cyl(s_0,(0,2],s_1)$

- $Pr_{s0}(Cyl(s_0,(0,2],s_1))$

$$= Pr_{s0}(Cyl(s_0)) \cdot \mathbf{P}^{emb(C)}(s_0,s_1) \cdot (e^{-E(s0)\cdot 0} - e^{-E(s0)\cdot 2})$$
$$= 1 \cdot \mathbf{P}^{emb(C)}(s_0,s_1) \cdot (e^{-E(s0)\cdot 0} - e^{-E(s0)\cdot 2})$$
$$= 1 \cdot 1 \cdot (e^{-3/2\cdot 0} - e^{-3/2\cdot 2})$$
$$= 1 - e^{-3}$$
$$\approx 0.95021$$

Probability space

- Probability space (Path(s), $\Sigma_{Path(s)}$, Pr_s)   (see [BHHK03])

- Sample space Ω = Path(s)
  - i.e. all infinite paths
- Event set $\Sigma_{Path(s)}$
  - least σ-algebra on Path(s) containing all cylinders sets $Cyl(s_0,I_0,\ldots,I_{n-1},s_n)$ where:
    - $s_0,\ldots,s_n$ ranges over all state sequences with $\mathbf{R}(s_i,s_{i+1})>0$ for all i
    - $I_0,\ldots,I_{n-1}$ ranges over all sequences of non-empty intervals in $\mathbb{R}_{\geq 0}$ (where intervals are bounded by rationals)
- Probability measure Pr_s
  - Pr_s extends uniquely from probability defined over cylinders

Probabilistic reachability

- Probabilistic reachability
  - the probability of reaching a target set $T \subseteq S$
  - measurability:
    - union of all basic cylinders $Cyl(s_0,(0,\infty),s_1,(0,\infty),\ldots,(0,\infty),s_n)$ where $s_n \in T$
    - set of such state sequences $s_0 s_1 \ldots s_n$ is countable

- Time-bounded probabilistic reachability
  - the probability of reaching a target set $T \subseteq S$ within t time units
  - measurability:
    - union of all basic cylinders $Cyl(s_0,I_0,s_1,I_1,\ldots,I_{n-1},s_n)$ where $s_n \in T$ and $sup(I_0)+\ldots+sup(I_{n-1}) \leq t$
    - set of such state sequences $s_0 s_1 \ldots s_n$ is countable
    - set of rational-bounded intervals is countable

Summing up…

- Exponential distribution
  - suitable for modelling failures, waiting times, reactions, …
  - nice mathematical properties
- Continuous-time Markov chains
  - transition delays modelled as exponential distributions
  - race condition
  - embedded DTMC
  - generator matrix
- Probability space over paths
  - (untimed and timed) probabilistic reachability