Lecture 3
Discrete-time Markov Chains...

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Next few lectures…

• Today:
  – Discrete–time Markov chains (continued)

• Mon 2pm:
  – Probabilistic temporal logics

• Wed 3pm:
  – PCTL model checking for DTMCs

• Thur 12pm:
  – PRISM
Overview

• Transient state probabilities

• Long-run / steady-state probabilities

• Qualitative properties
  – repeated reachability
  – persistence
Transient state probabilities

- What is the probability, having started in state $s$, of being in state $s'$ at time $k$?
  - i.e. after exactly $k$ steps/transitions have occurred
  - this is the transient state probability: $\pi_{s,k}(s')$

- Transient state distribution: $\pi_{s,k}$
  - vector $\pi_{s,k}$ i.e. $\pi_{s,k}(s')$ for all states $s'$

- Note: this is a discrete probability distribution
  - so we have $\pi_{s,k} : S \rightarrow [0,1]$
  - rather than e.g. $\Pr_s : \Sigma_{\text{Path}(s)} \rightarrow [0,1]$ where $\Sigma_{\text{Path}(s)} \subseteq 2^{\text{Path}(s)}$
Transient distributions

$k=0$:

$k=1$:

$k=2$:

$k=3$:
Computing transient probabilities

• Transient state probabilities:
  - $\pi_{s,k}(s') = \sum_{s'' \in S} P(s'',s') \cdot \pi_{s,k-1}(s'')$
  - (i.e. look at incoming transitions)

• Computation of transient state distribution:
  - $\pi_{s,0}$ is the initial probability distribution
  - e.g. in our case $\pi_{s,0}(s') = 1$ if $s' = s$ and $\pi_{s,0}(s') = 0$ otherwise
  - $\pi_{s,k} = \pi_{s,k-1} \cdot P$

• i.e. successive vector–matrix multiplications
Computing transient probabilities

\[
P = \begin{bmatrix}
0 & 0.5 & 0 & 0.5 & 0 & 0 \\
0.5 & 0 & 0.25 & 0 & 0.25 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\pi_{s0,0} = \begin{bmatrix} 1, 0, 0, 0, 0, 0 \end{bmatrix}
\]

\[
\pi_{s0,1} = \begin{bmatrix} 0, \frac{1}{2}, 0, \frac{1}{2}, 0, 0 \end{bmatrix}
\]

\[
\pi_{s0,2} = \begin{bmatrix} \frac{1}{4}, 0, \frac{1}{8}, \frac{1}{2}, \frac{1}{8}, 0 \end{bmatrix}
\]

\[
\pi_{s0,3} = \begin{bmatrix} 0, \frac{1}{8}, 0, \frac{5}{8}, \frac{1}{8}, \frac{1}{8} \end{bmatrix}
\]

...
Computing transient probabilities

- \( \pi_{s,k} = \pi_{s,k-1} \cdot P = \pi_{s,0} \cdot P^k \)

- \( k^{th} \) matrix power: \( P^k \)
  - \( P \) gives one-step transition probabilities
  - \( P^k \) gives probabilities of \( k \)-step transition probabilities
  - i.e. \( P^k(s,s') = \pi_{s,k}(s') \)

- A possible optimisation: iterative squaring
  - e.g. \( P^8 = ((P^2)^2)^2 \)
  - only requires \( \log k \) multiplications
  - but potentially inefficient, e.g. if \( P \) is large and sparse

  - in practice, successive vector–matrix multiplications preferred
Notion of time in DTMCs

- Two possible views on the timing aspects of a system modelled as a DTMC:

  - Discrete time-steps model time accurately
    - e.g. clock ticks in a model of an embedded device
    - or like dice example: interested in number of steps (tosses)

  - Time-abstract
    - no information assumed about the time transitions take
    - e.g. simple Zeroconf model

- In the latter case, transient probabilities are not very useful
- In both cases, often beneficial to study long-run behaviour
Long-run behaviour

- **Consider the limit:** \( \pi_s = \lim_{k \to \infty} \pi_{s,k} \)
  - where \( \pi_{s,k} \) is the transient state distribution at time \( k \) having starting in state \( s \)
  - this limit, where it exists, is called the limiting distribution

- **Intuitive idea**
  - the percentage of time, in the long run, spent in each state
  - e.g. reliability: “in the long–run, what percentage of time is the system in an operational state”
Limiting distribution

- Example:

\[
\pi_{s0,0} = \begin{bmatrix} 1,0,0,0,0,0 \end{bmatrix}
\]
\[
\pi_{s0,1} = \begin{bmatrix} 0,\frac{1}{2},0,\frac{1}{2},0,0 \end{bmatrix}
\]
\[
\pi_{s0,2} = \begin{bmatrix} \frac{1}{4},0,\frac{1}{8},\frac{1}{2},\frac{1}{8},0 \end{bmatrix}
\]
\[
\pi_{s0,3} = \begin{bmatrix} 0,\frac{1}{8},0,\frac{5}{8},\frac{1}{8},\frac{1}{8} \end{bmatrix}
\]
\[
\ldots
\]
\[
\pi_{s0} = \begin{bmatrix} 0,0,\frac{1}{12},\frac{2}{3},\frac{1}{6},\frac{1}{12} \end{bmatrix}
\]
Long-run behaviour

• Questions:
  – when does this limit exist?
  – does it depend on the initial state/distribution?

Need to consider underlying graph
  – \((V, E)\) where \(V\) are vertices and \(E \subseteq V \times V\) are edges
  – \(V = S\) and \(E = \{ (s, s') \text{ s.t. } P(s, s') > 0 \}\)
Graph terminology

• A state $s'$ is **reachable** from $s$ if there is a finite path starting in $s$ and ending in $s'$

• A subset $T$ of $S$ is **strongly connected** if, for each pair of states $s$ and $s'$ in $T$, $s'$ is reachable from $s$ passing only through states in $T$

• A **strongly connected component (SCC)** is a maximally strongly connected set of states (i.e. no superset of it is also strongly connected)

• A **bottom strongly connected component (BSCC)** is an SCC $T$ from which no state outside $T$ is reachable from $T$

• Alternative terminology: “$s$ communicates with $s'$”, “communicating class”, “closed communicating class”
Example – (B)SCCs

SCC

BSCC

BSCC

BSCC

BSCC

DP/Probabilistic Model Checking, Michaelmas 2011
Graph terminology

• Markov chain is **irreducible** if all its states belong to a single BSCC; otherwise reducible

  ![Diagram](image)

  - **S₀**
  - **S₁**

  1

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• A state **s** is **periodic**, with period **d**, if
  
  – the greatest common divisor of the set \( \{ n \mid f_s^{(n)} > 0 \} \) equals \( d \)
  
  – where \( f_s^{(n)} \) is the probability of, when starting in state \( s \), returning to state \( s \) in exactly \( n \) steps

• A Markov chain is **aperiodic** if its period is 1
Steady-state probabilities

- For a finite, irreducible, aperiodic DTMC...
  - limiting distribution always exists
  - and is independent of initial state/distribution

- These are known as steady-state probabilities
  - (or equilibrium probabilities)
  - effect of initial distribution has disappeared, denoted $\pi$

- These probabilities can be computed as the unique solution of the linear equation system:

$$\pi \cdot P = \pi \text{ and } \sum_{s \in S} \pi(s) = 1$$
Steady-state – Balance equations

- Known as *balance equations*

\[
\pi \cdot P = \pi \quad \text{and} \quad \sum_{s \in S} \pi(s) = 1
\]

- That is:

\[
- \pi(s') = \sum_{s \in S} \pi(s) \cdot P(s, s')
\]

\[
- \sum_{s \in S} \pi(s) = 1
\]

balance the probability of leaving and entering a state \(s'\)

normalisation
Let $\mathbf{x} = \pi$

Solve: $\mathbf{x} \cdot \mathbf{P} = \mathbf{x}$, $\sum_s \mathbf{x}(s) = 1$

\[
\mathbf{P} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0.01 & 0.01 & 0.98 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]

$\mathbf{x} \approx [0.332215, 0.335570, 0.003356, 0.328859]$
### Steady-state – Example

- Let $\mathbf{x} = \pi$
- Solve: $\mathbf{x} \cdot \mathbf{P} = \mathbf{x}$, $\sum_s x(s) = 1$

\[
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0.01 & 0.01 & 0.98 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]

\[x \approx [ 0.332215, 0.335570, 0.003356, 0.328859 ]\]

Long-run percentage of time spent in the state “try”
\[\approx 33.6\%\]

Long-run percentage of time spent in “fail”/”succ”
\[\approx 0.003356 + 0.328859 \approx 33.2\%\]
Periodic DTMCs

- For (finite, irreducible) periodic DTMCs, this limit:
  \[ \pi_s(s') = \lim_{k \to \infty} \pi_{s,k}(s') \]

- does not exist, but this limit does:
  \[ \lim_{n \to \infty} \frac{1}{n} \cdot \sum_{k=1}^{n} \pi_{s,k}(s') \]
  (and where both limits exist, e.g. for aperiodic DTMCs, these 2 limits coincide)

- Steady-state probabilities for these DTMCs can be computed by solving the same set of linear equations:
  \[ \pi \cdot P = \pi \text{ and } \sum_{s \in S} \pi(s) = 1 \]
Steady-state – General case

- **General case: reducible DTMC**
  - compute vector \( \pi_s \)
  - (note: distribution depends on initial state \( s \))
- **Compute BSCCs for DTMC; then two cases to consider:**
  - (1) \( s \) is in a BSCC \( T \)
    - compute steady-state probabilities \( x \) in sub-DTMC for \( T \)
    - \( \pi_s(s') = x(s') \) if \( s' \) in \( T \)
    - \( \pi_s(s') = 0 \) if \( s' \) not in \( T \)
  - (2) \( s \) is not in any BSCC
    - compute steady-state probabilities \( x_T \) for sub-DTMC of each BSCC \( T \) and combine with reachability probabilities to BSCCs
    - \( \pi_s(s') = \text{ProbReach}(s, T) \cdot x_T(s') \) if \( s' \) is in BSCC \( T \)
    - \( \pi_s(s') = 0 \) if \( s' \) is not in a BSCC
Steady-state – Example 2

- $\pi_s$ depends on initial state $s$

\[
\begin{align*}
\pi_{s0} &= \begin{bmatrix} 0,0,0,0,0,0 \end{bmatrix} \\
\pi_{s1} &= \ldots \\
\pi_{s2} &= \pi_{s5} = \begin{bmatrix} 0,0,\frac{1}{2},0,0,\frac{1}{2} \end{bmatrix} \\
\pi_{s3} &= \begin{bmatrix} 0,0,1,0,0,0 \end{bmatrix} \\
\pi_{s4} &= \begin{bmatrix} 0,0,0,0,1,0 \end{bmatrix} \\
\pi_{s5} &= \begin{bmatrix} 0,0,0,0,0,0 \end{bmatrix}
\end{align*}
\]
Qualitative properties

- **Quantitative properties:**
  - “what is the probability of event A?”

- **Qualitative properties:**
  - “the probability of event A is 1” (“almost surely A”)  
  - or: “the probability of event A is > 0” (“possibly A”)

- For finite DTMCs, qualitative properties do not depend on the transition probabilities – only need underlying graph
  - e.g. to determine “is target set T reached with probability 1?” (see DTMC model checking lecture)
  - computing BSCCs of a DTMCs yields information about long-run qualitative properties...
Fundamental property

- Fundamental property of (finite) DTMCs...

- With probability 1, a BSCC will be reached and all of its states visited infinitely often

- Formally:
  \[
  \Pr_{s_0}(s_0 s_1 s_2 \ldots \mid \exists i \geq 0, \exists \text{ BSCC } T \text{ such that } \\
  \forall j \geq i s_j \in T \text{ and } \\
  \forall s \in T s_k = s \text{ for infinitely many } k) = 1
  \]
Zeroconf example

- 2 BSCCs: \{s_6\}, \{s_8\}
- Probability of trying to acquire a new address infinitely often is 0
Aside: Infinite Markov chains

- Infinite-state random walk

\[ \text{Value of probability } p \text{ does affect qualitative properties} \]

- \( \text{ProbReach}(s, \{s_0\}) = 1 \text{ if } p \leq 0.5 \)

- \( \text{ProbReach}(s, \{s_0\}) < 1 \text{ if } p > 0.5 \)
Repeated reachability

- Repeated reachability:
  - “always eventually…”, “infinitely often…”
- $\Pr_{s_0} ( s_0 s_1 s_2 \ldots \mid \forall \ i \geq 0 \ \exists \ j \geq i \ s_j \in B )$
  - where $B \subseteq S$ is a set of states

- e.g. “what is the probability that the protocol successfully sends a message infinitely often?”

- Is this measurable? Yes…
  - set of satisfying paths is: $\bigcap_{n \geq 0} \bigcup_{m \geq n} C_m$
  - where $C_m$ is the union of all cylinder sets $\text{Cyl}(s_0 s_1 \ldots s_m)$ for finite paths $s_0 s_1 \ldots s_m$ such that $s_m \in B$
Qualitative repeated reachability

- $\Pr_{s_0}(s_0s_1s_2... \mid \forall \ i \geq 0 \ \exists \ j \geq i \ s_j \in B) = 1$
- $\Pr_{s_0}(\text{"always eventually } B\text{"}) = 1$

if and only if

- $T \cap B \neq \emptyset$ for each BSCC $T$ that is reachable from $s_0$

Example:
$B = \{ s_3, s_4, s_5 \}$
Persistence

• Persistence properties:
  – “eventually forever…”

• $\Pr_{s_0} ( s_0 s_1 s_2 \ldots | \exists \ i \geq 0 \ \forall \ j \geq i \ s_j \in B )$
  – where $B \subseteq S$ is a set of states

• e.g. “what is the probability of the leader election algorithm reaching, and staying in, a stable state?”

• e.g. “what is the probability that an irrecoverable error occurs?”

• Is this measurable? Yes…
Qualitative persistence

• \( \Pr_{s_0}( s_0s_1s_2... \mid \exists i \geq 0 \ \forall j \geq i \ s_j \in B ) = 1 \)

\( \Pr_{s_0}( \text{“eventually forever } B \text{”} ) = 1 \)

if and only if

• \( T \subseteq B \) for each BSCC \( T \) that is reachable from \( s_0 \)

Example:

\( B = \{ s_2, s_3, s_4, s_5 \} \)
Summing up…

- **Transient state probabilities**
  - successive vector–matrix multiplications

- **Long–run/steady–state probabilities**
  - requires graph analysis
  - irreducible case: solve linear equation system
  - reducible case: steady–state for sub–DTMCs + reachability

- **Qualitative properties**
  - repeated reachability
  - persistence