Written Exercises

These written exercises are based around the material on discrete-time Markov chains (DTMCs) and Markov decision processes (MDPs) from the course "Advances in Probabilistic Model Checking".

See the course web site for links to useful resources:

- `http://www.prismmodelchecker.org/courses/marktoberdorf11/`

You can also find a practical exercise, based on the PRISM tool, at the address above.

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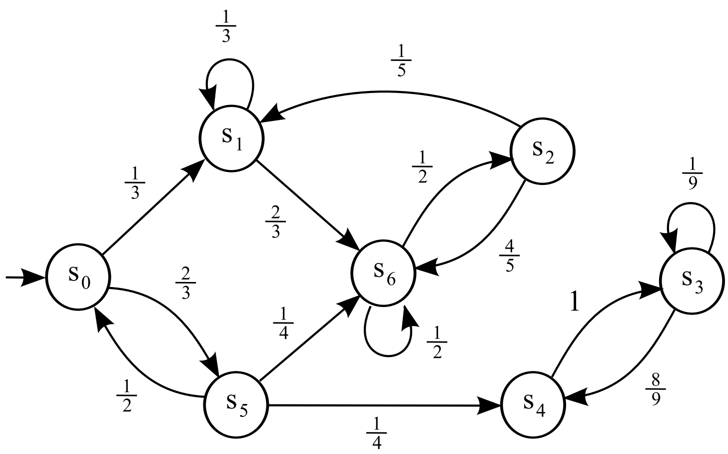
1. Consider the DTMC below:



Let $A = \{s_3\}$ and $B = \{s_2\}$.

(a) Compute the probability measure of the *union* of the following cylinder sets: $\text{Cyl}(s_0s_1)$, $\text{Cyl}(s_0s_5s_6)$, $\text{Cyl}(s_0s_5s_4s_3)$, $\text{Cyl}(s_0s_1s_6)$

(b) Give a PCTL formula stating that the probability of reaching a state in A is at least $\frac{1}{5}$.

(c) Compute the probability, from each state of the Markov chain, of reaching a state in A.

(d) What is the probability, from the initial state, of reaching the set of states $A \cup B$?

(e) What is the probability, from the initial state, that a state from $A \cup B$ is visited infinitely often?

2. Consider the following simple game between 2 players who, between them, have n coins. Initially, player 1 has m of these coins. In each turn of the game, both players simultaneously toss one of their coins. If the two coins are the same, player 1 keeps both coins; if they differ, player 2 keeps them. The game ends when one player has all the coins and is declared the winner.

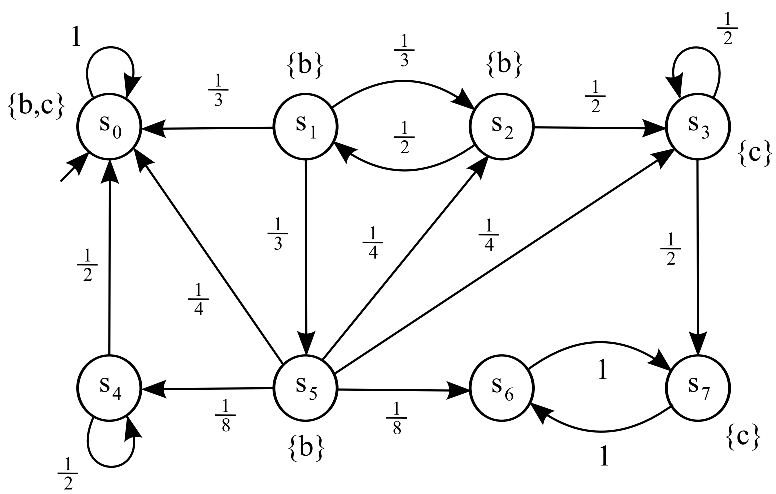
   (a) Draw a DTMC to represent the evolution of this game.

   (b) What is the probability of the game terminating?

   (c) Assume we wish to establish if "player 1 has a better chance of winning than player 2". Express this statement using PCTL.

   (d) Express the following statement in PCTL: "with probability at most 0.1, player 1 will, within 5 more turns, be in a position where he has a chance to win the game in the next turn".

3. Consider the DTMC below. Illustrate the execution of the PCTL model checking algorithms to determine which states of the Markov chain satisfy $\mathrm{P}_{\geq \frac{17}{19}}[\, b \, \mathrm{U} \, c \,]$.

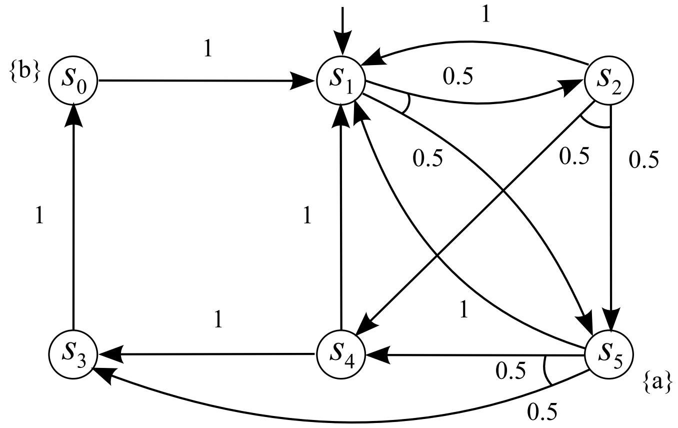


4. Consider two parallel processes, A and B, each of which can be in 3 states, say $\{0, 1, 2\}$, and is initially in state 0. Periodically, each process can (independently) receive a request that requires action. When this occurs, a process moves to state 1. A third process C will observe that process A or B is in state 1 and take the request to be executed. When this occurs, A or B moves to state 2. When A and B are both in state 1, process C selects either A or B at random (with equal probability). From state 2, A or B moves back to state 0.

   (a) Draw an MDP with state space $\{0, 1, 2\} \times \{0, 1, 2\}$ representing the system described above.

   (b) Give a PCTL formula to express the statement "if either of the two processes is in state 1, then with probability 1, both processes will eventually be simultaneously in state 0". Is this statement true? Justify your answer.

5. This question concerns the following MDP:



   (a) Execute the PCTL model checking algorithm to determine which states of the MDP satisfy the PCTL formulae $\mathrm{P}_{\leq 0.3}[\,\neg a \,\mathrm{U}\, b\,]$.

   (b) For the same PCTL property, deduce a memoryless adversary which results in the computed probabilities and give the corresponding DTMC.

6. Show whether each of the two PCTL formulae below is *satisfiable*. That is, either provide an example of an MDP for which at least one state satisfies the formula, or prove that this is impossible.

   (a) $\mathrm{P}_{>0.5}[\,\mathrm{X}\, a\,] \wedge \mathrm{P}_{<0.5}[\,\mathrm{F}\, a\,]$

   (b) $\neg\mathrm{P}_{\leq 0.6}[\,\mathrm{X}\, a\,] \wedge \neg\mathrm{P}_{\geq 0.4}[\,\mathrm{F}\, a\,]$