Probabilistic model checking with PRISM

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Lecture plan

- Course slides and lab session
  - http://www.prismmodelchecker.org/courses/imt16/
- 3 sessions: lectures 9-11
  - 1 – Discrete time Markov chains (DTMCs)
  - 2 – Markov decision processes (MDPs)
  - 3 – LTL model checking for DTMCs/MDPs
- For extended versions of this material
  - and an accompanying list of references
  - see: http://www.prismmodelchecker.org/lectures/

Probabilistic models

	Fully probabilistic	Nondeterministic
Discrete time	Discrete-time Markov chains (DTMCs)	Markov decision processes (MDPs) Simple stochastic games (SMGs)
Continuous time	Continuous-time Markov chains (CTMCs)	Probabilistic timed automata (PTAs) Interactive Markov chains (IMCs)

Part 2

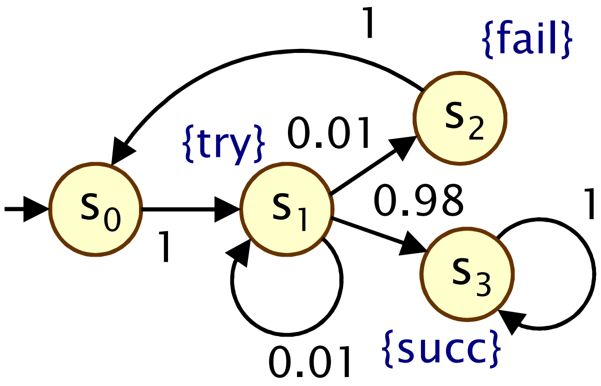
Markov decision processes

Overview (Part 2)

- Introduction
- Model checking for Markov decision processes (MDPs)
  - MDPs: definition
  - Paths, strategies & probability spaces
  - PCTL model checking
  - Costs and rewards
  - Case study: Firewire root contention
- Strategy synthesis for MDPs
  - Properties and objectives
  - Verification vs synthesis
  - Case study: Dynamic power management
- Summary

Recap: Discrete-time Markov chains

- Discrete-time Markov chains (DTMCs)
  - state-transition systems augmented with probabilities
- Formally: DTMC D = (S, s_{init}, P, L) where:
  - S is a set of states and $s_{init} \in S$ is the initial state
  - $P : S \times S \rightarrow [0,1]$ is the transition probability matrix
  - $L : S \rightarrow 2^{AP}$ labels states with atomic propositions
  - define a probability space Pr_s over paths $Path_s$



- Properties of DTMCs
  - can be captured by the logic PCTL
  - e.g. send → $P_{\geq 0.95}$ [F deliver]
  - key question: what is the probability of reaching states $T \subseteq S$ from state s?
  - reduces to graph analysis + linear equation system

Nondeterminism

- Some aspects of a system may not be probabilistic and should not be modelled probabilistically; for example:

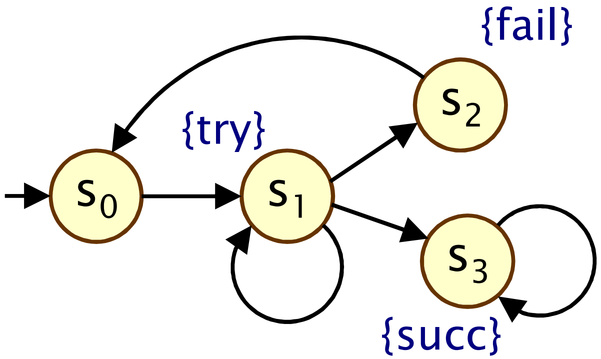
- Concurrency – scheduling of parallel components
    - e.g. randomised distributed algorithms – multiple probabilistic processes operating asynchronously

- Underspecification – unknown model parameters
    - e.g. a probabilistic communication protocol designed for message propagation delays of between d_{min} and d_{max}

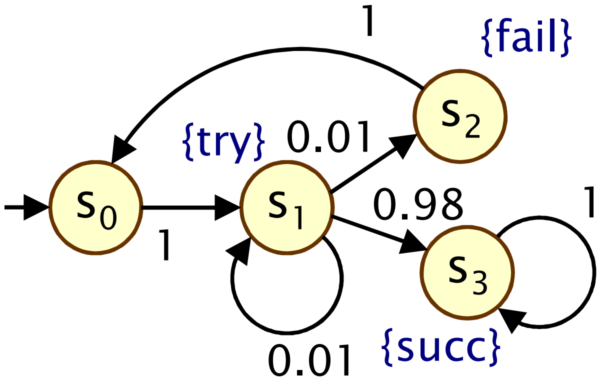
- Unknown environments
    - e.g. probabilistic security protocols – unknown adversary

Probability vs. nondeterminism

- Labelled transition system
  - (S,s_0,R,L) where $R \subseteq S \times S$
  - choice is nondeterministic



- Discrete-time Markov chain
  - (S,s_0,P,L) where $P : S \times S \rightarrow [0,1]$
  - choice is probabilistic



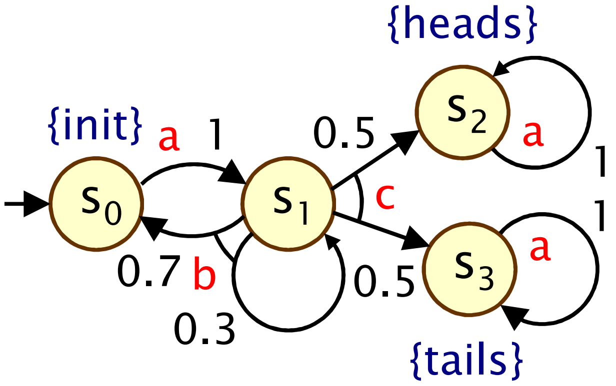
- How to combine?

Markov decision processes

- Markov decision processes (MDPs)
  - extension of DTMCs which allow nondeterministic choice

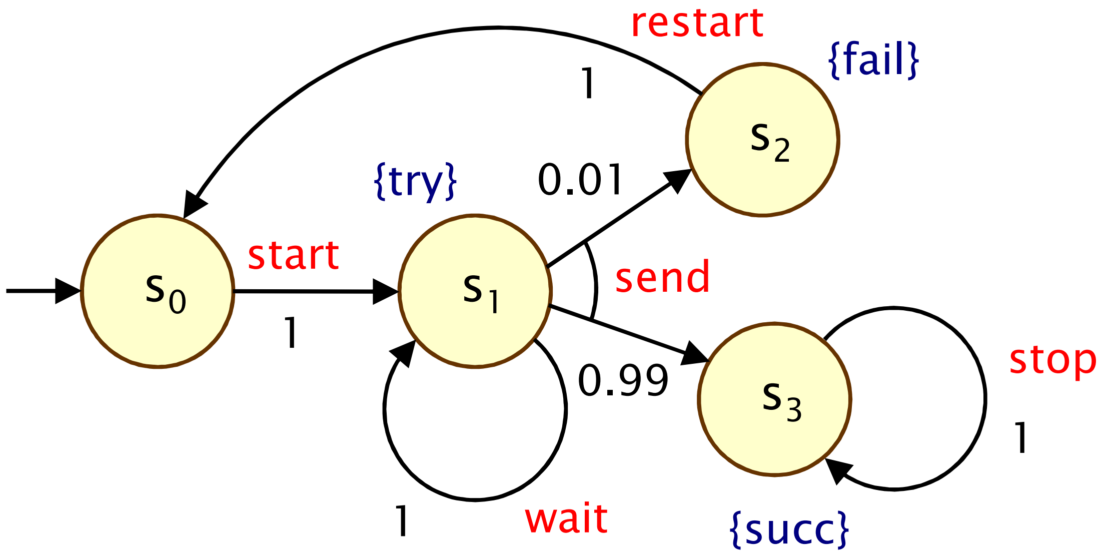
- Like DTMCs:
  - discrete set of states representing possible configurations of the system being modelled
  - transitions between states occur in discrete time-steps

- Probabilities and nondeterminism
  - in each state, a nondeterministic choice between several discrete probability distributions over successor states



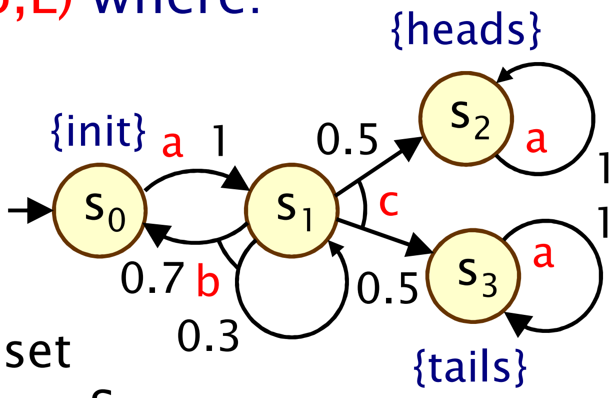
Simple MDP example

- A simple communication protocol
  - after one step, process starts trying to send a message
  - then, a nondeterministic choice between: (a) waiting a step because the channel is unready; (b) sending the message
  - if the latter, with probability 0.99 send successfully and stop
  - and with probability 0.01, message sending fails, restart



Markov decision processes

- Formally, an MDP M is a tuple $(S,s_{init},\alpha,\delta,L)$ where:
  - S is a set of states ("state space")
  - $s_{init} \in S$ is the initial state
  - α is an alphabet of action labels
  - $\delta \subseteq S \times \alpha \times Dist(S)$ is the transition probability relation, where Dist(S) is the set of all discrete probability distributions over S
  - $L : S \rightarrow 2^{AP}$ is a labelling with atomic propositions



- Notes:
  - we also abuse notation and use δ as a function
  - i.e. $\delta : S \rightarrow 2^{\alpha \times Dist(S)}$ where $\delta(s) = \{ (a,\mu) \mid (s,a,\mu) \in \delta \}$
  - we assume $\delta(s)$ is always non-empty, i.e. no deadlocks
  - MDPs, here, are identical to probabilistic automata [Segala]
    - usually, MDPs take the form: $\delta : S \times \alpha \rightarrow Dist(S)$

Simple MDP example 2

$M = (S, s_{init}, Steps, L)$

$S = \{s_0, s_1, s_2, s_3\}$

$s_{init} = s_0$

$AP = \{init, heads, tails\}$

$L(s_0)=\{init\}$,

$L(s_1)=\varnothing$,

$L(s_2)=\{heads\}$,

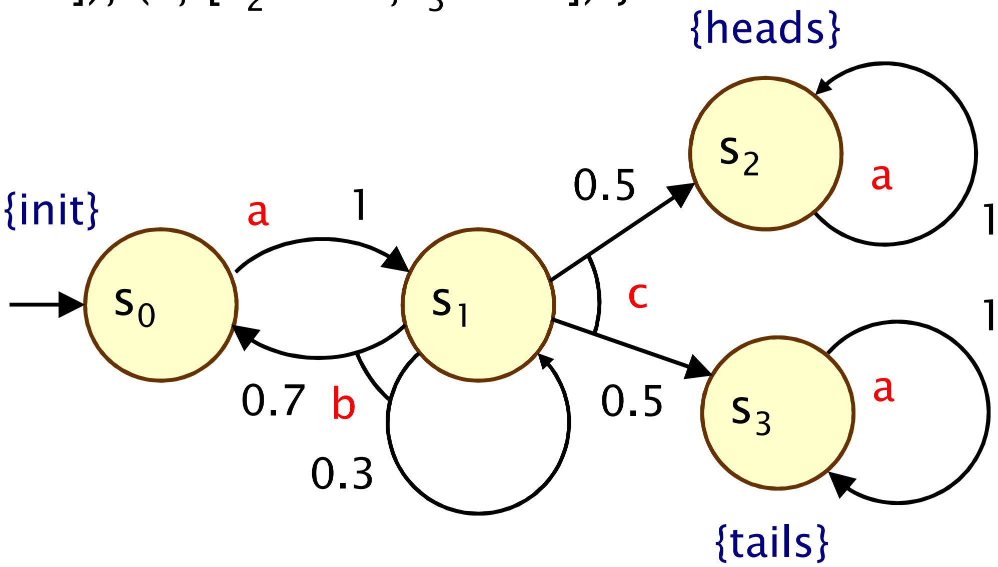
$L(s_3)=\{tails\}$

$Steps(s_0) = \{ (a, [s_1 \mapsto 1]) \}$

$Steps(s_1) = \{ (b, [s_0 \mapsto 0.7, s_1 \mapsto 0.3]), (c, [s_2 \mapsto 0.5, s_3 \mapsto 0.5]) \}$

$Steps(s_2) = \{ (a, [s_2 \mapsto 1]) \}$

$Steps(s_3) = \{ (a, [s_3 \mapsto 1]) \}$



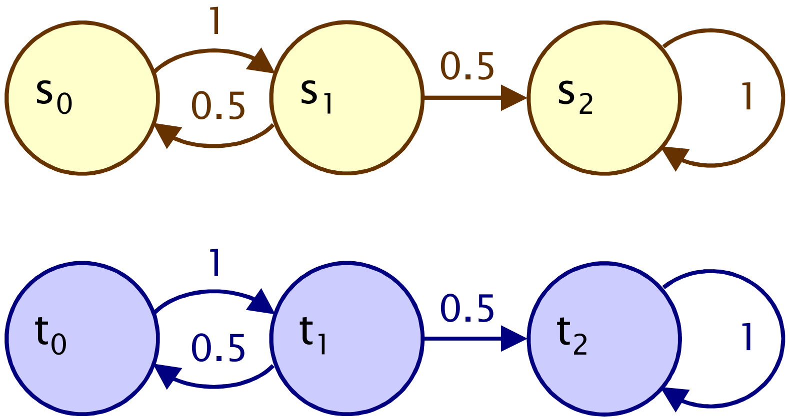
Example – Parallel composition

Asynchronous parallel composition of two 3-state DTMCs

PRISM code:

```
module M1
    s : [0..2] init 0;
    [] s=0 -> (s'=1);
    [] s=1 -> 0.5:(s'=0) + 0.5:(s'=2);
    [] s=2 -> (s'=2);
endmodule
```

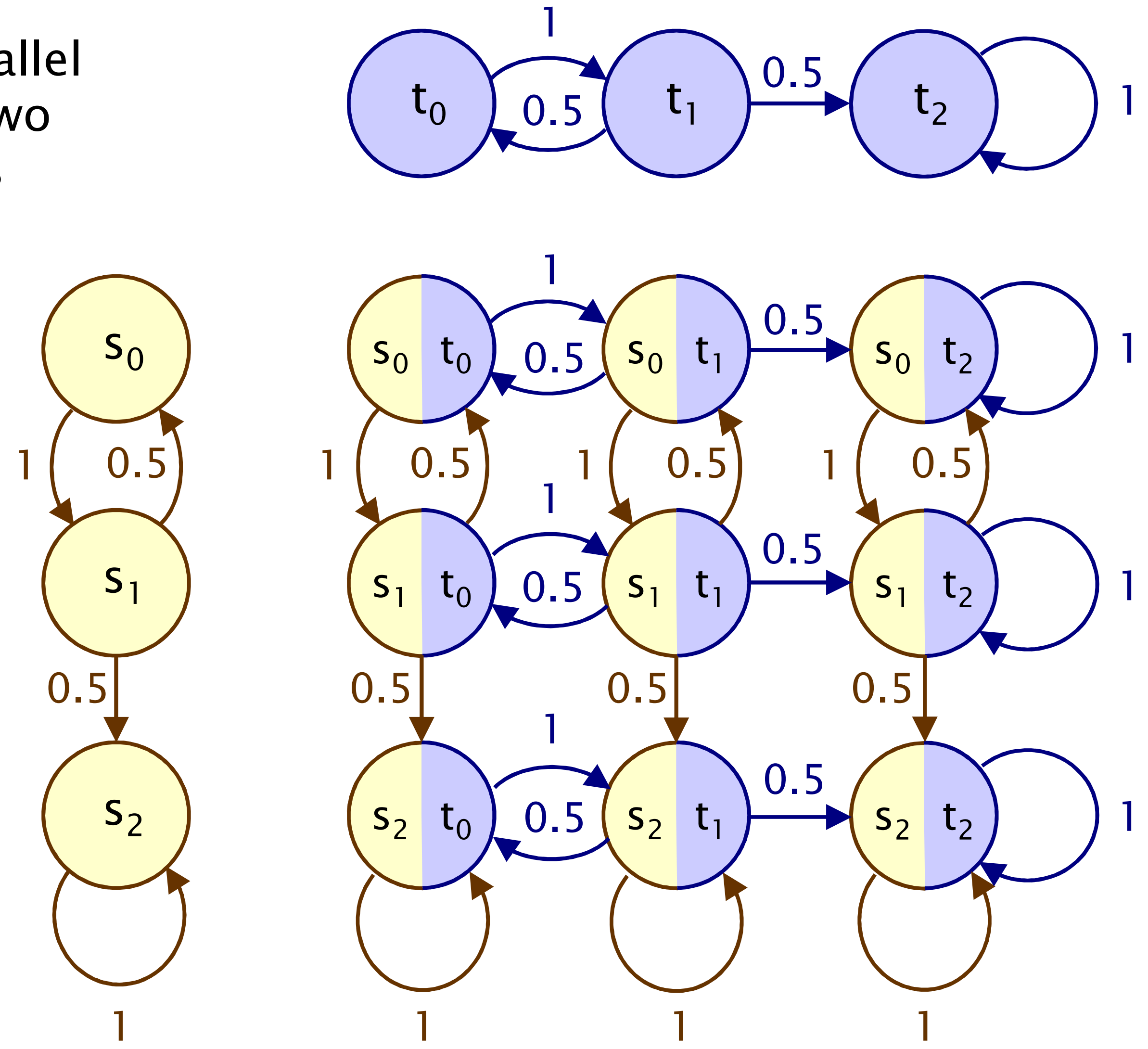
```
module M2 = M1 [ s=t ] endmodule
```



Example – Parallel composition

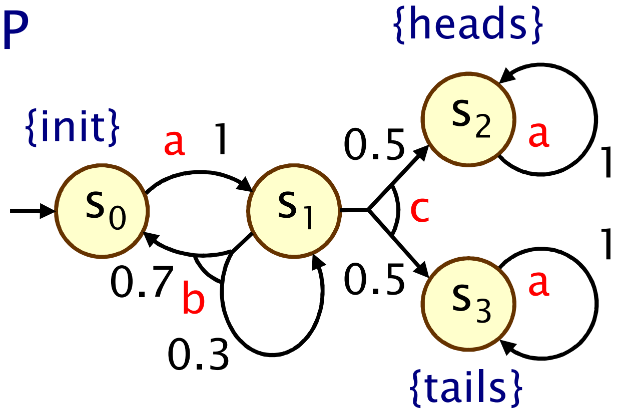
Asynchronous parallel composition of two 3-state DTMCs

Action labels omitted here



Paths and strategies

- A (finite or infinite) path through an MDP
  - is a sequence ($s_0 \ldots s_n$) of (connected) states
  - represents an execution of the system
  - resolves both the probabilistic and nondeterministic choices



- A strategy σ (aka. "adversary" or "policy") of an MDP
  - is a resolution of nondeterminism only
  - is (formally) a mapping from finite paths to distributions on action-distribution pairs
  - induces a fully probabilistic model
  - i.e. an (infinite-state) Markov chain over finite paths
  - on which we can define a probability space over infinite paths

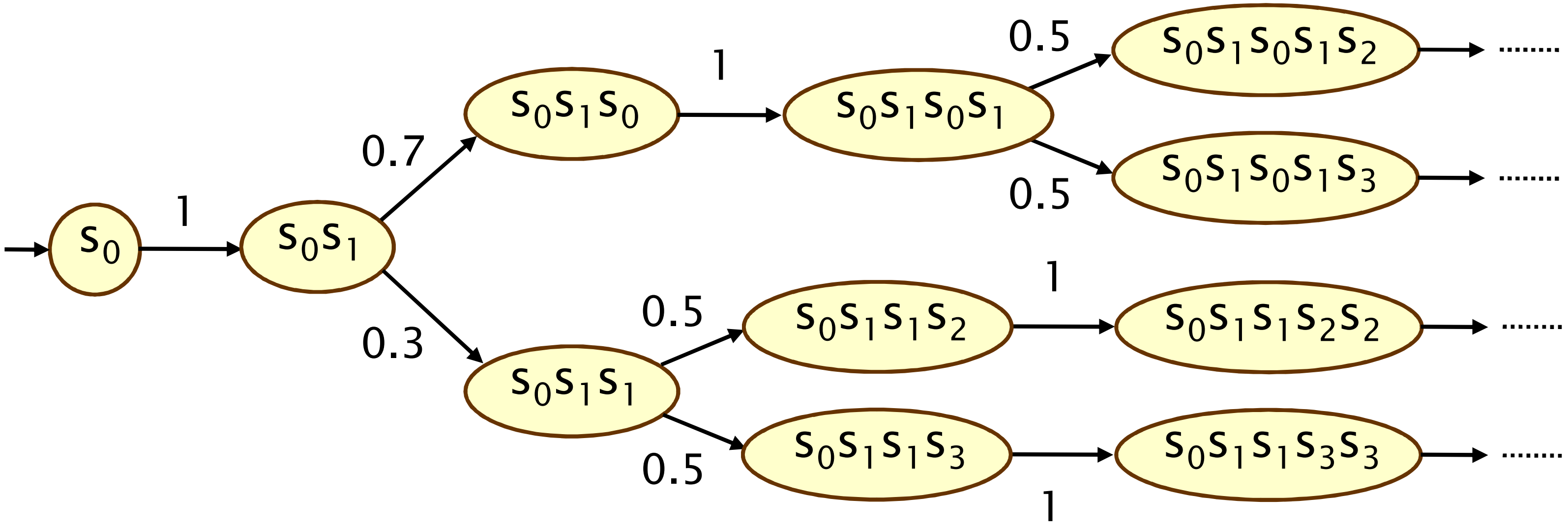
Classification of strategies

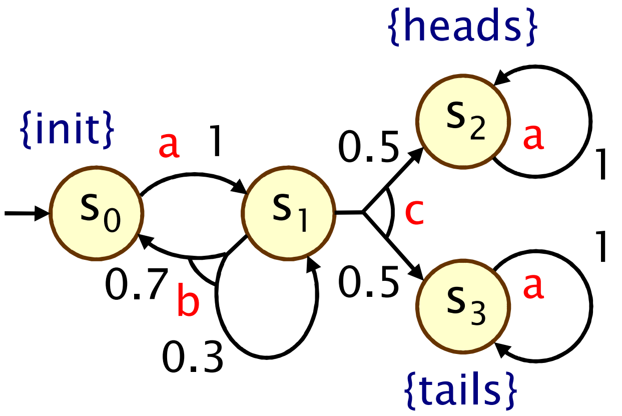
- Strategies are classified according to
- randomisation:
  - σ is deterministic (pure) if $\sigma(s_0 \ldots s_n)$ is a point distribution, and randomised otherwise
- memory:
  - σ is memoryless (simple) if $\sigma(s_0 \ldots s_n) = \sigma(s_n)$ for all $s_0 \ldots s_n$
  - σ is finite memory if there are finitely many modes such as $\sigma(s_0 \ldots s_n)$ depends only on s_n and the current mode, which is updated each time an action is performed
  - otherwise, σ is infinite memory

- A strategy σ induces, for each state s in the MDP:
  - a set of infinite paths $Path^{\sigma}(s)$
  - a probability space Pr^{σ}_{s} over $Path^{\sigma}(s)$

Example strategy

- Fragment of induced Markov chain for strategy which picks b then c in s_1





finite-memory,
deterministic

Induced DTMCs

- Strategy σ for MDP induces an infinite-state DTMC D^σ

- $D^\sigma = (Path^\sigma_{fin}(s), s, P^\sigma_s)$ where:
  - states of the DTMC are the finite paths of σ starting in state s
  - initial state is s (the path starting in s of length 0)
  - $P^\sigma_s(\omega,\omega')=\mu(s')$ if $\omega'= \omega(a, \mu)s'$ and $\sigma(\omega)=(a,\mu)$
  - $P^\sigma_s(\omega,\omega')=0$ otherwise

- 1-to-1 correspondence between $Path^\sigma(s)$ and paths of D^σ

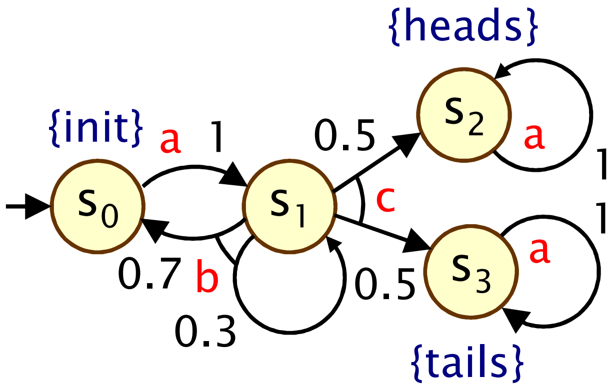
- This gives us a probability measure Pr^σ_s over $Path^\sigma(s)$
  - from probability measure over paths of D^σ

MDPs and probabilities

- $\mathrm{Prob}^{\sigma}(s, \psi) = \mathrm{Pr}^{\sigma}_{s} \{ \omega \in \mathrm{Path}^{\sigma}(s) \mid \omega \vDash \psi \}$
  - for some path formula ψ
  - e.g. $\mathrm{Prob}^{\sigma}(s, \mathrm{F}\ \mathrm{tails})$
- MDP provides best-/worst-case analysis
  - based on lower/upper bounds on probabilities
  - over all possible adversaries

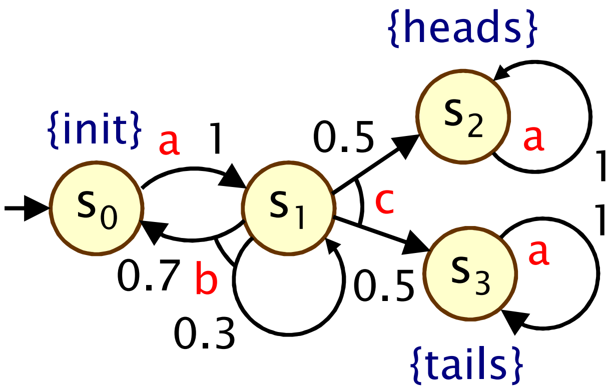
$$p_{\min}(s,\psi) = \inf_{\sigma \in Adv} \mathrm{Prob}^{\sigma}(s,\psi)$$

$$p_{\max}(s,\psi) = \sup_{\sigma \in Adv} \mathrm{Prob}^{\sigma}(s,\psi)$$

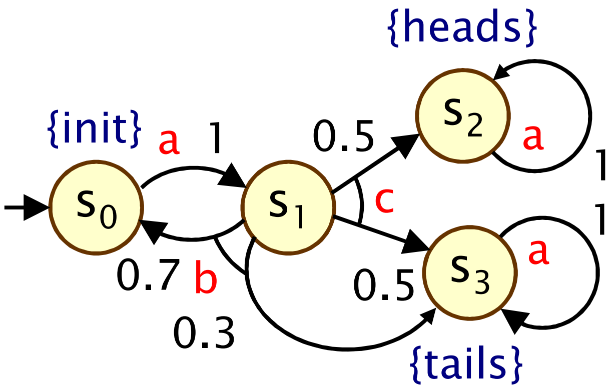


Examples

- $\mathrm{Prob}^{\sigma_1}(s_0, \mathrm{F\ tails}) = 0.5$
- $\mathrm{Prob}^{\sigma_2}(s_0, \mathrm{F\ tails}) = 0.5$
  - (where σ_i picks b i−1 times then c)
- …
- $p_{max}(s_0, \mathrm{F\ tails}) = 0.5$
- $p_{min}(s_0, \mathrm{F\ tails}) = 0$

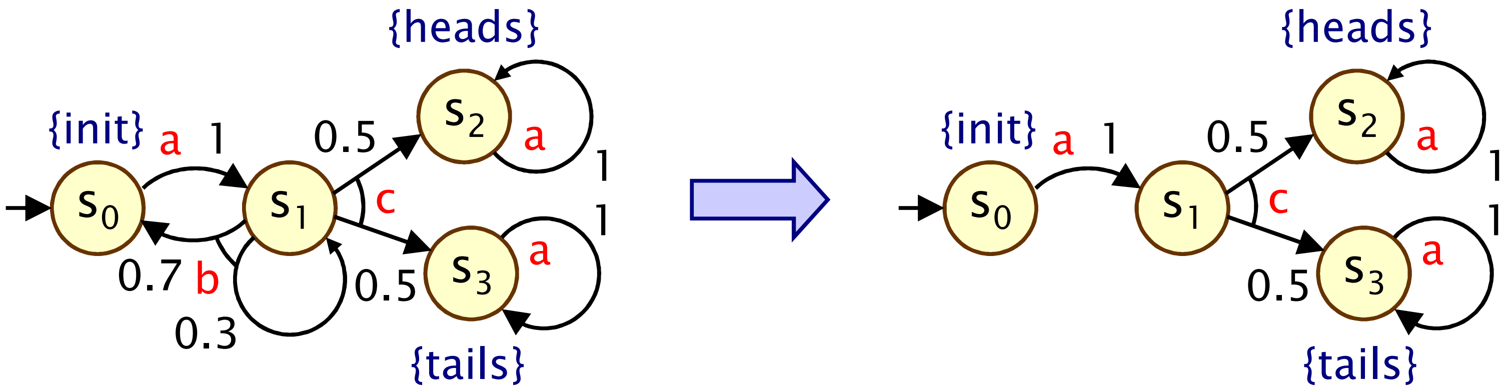


- $\mathrm{Prob}^{\sigma_1}(s_0, \mathrm{F\ tails}) = 0.5$
- $\mathrm{Prob}^{\sigma_2}(s_0, \mathrm{F\ tails})$
  $= 0.3+0.7\cdot0.5 = 0.65$
- $\mathrm{Prob}^{\sigma_3}(s_0, \mathrm{F\ tails})$
  $= 0.3+0.7\cdot0.3+0.7\cdot0.7\cdot0.5 = 0.755$
- …
- $p_{max}(s_0, \mathrm{F\ tails}) = 1$
- $p_{min}(s_0, \mathrm{F\ tails}) = 0.5$



Memoryless strategies

- Memoryless strategies always pick same choice in a state
  - also known as: positional, Markov, simple
  - formally, $\sigma(s_0(a_0,\mu_0)s_1\ldots s_n)$ depends only on s_n
  - can write as a mapping from states, i.e. $\sigma(s)$ for each $s \in S$
  - induced DTMC can be mapped to a $|S|$-state DTMC
- From previous example:
  - adversary σ_1 (picks c in s_1) is memoryless; σ_2 is not



PCTL

- Temporal logic for properties of MDPs (and DTMCs)
  - extension of (non-probabilistic) temporal logic CTL
  - key addition is probabilistic operator P
  - quantitative extension of CTL's A and E operators

- PCTL syntax:
  - $\phi ::= \text{true} \mid a \mid \phi \wedge \phi \mid \neg\phi \mid P_{\sim p} [\ \psi\ ]$ (state formulas)
  - $\psi ::= X\ \phi \mid \phi\ U^{\leq k}\ \phi \mid \phi\ U\ \phi$ (path formulas)
  - where a is an atomic proposition, used to identify states of interest, $p \in [0,1]$ is a probability, $\sim \in \{<,>,\leq,\geq\}$, $k \in \mathbb{N}$

- Example: send $\rightarrow P_{\geq 0.95}$ [true $U^{\leq 10}$ deliver]

PCTL semantics for MDPs

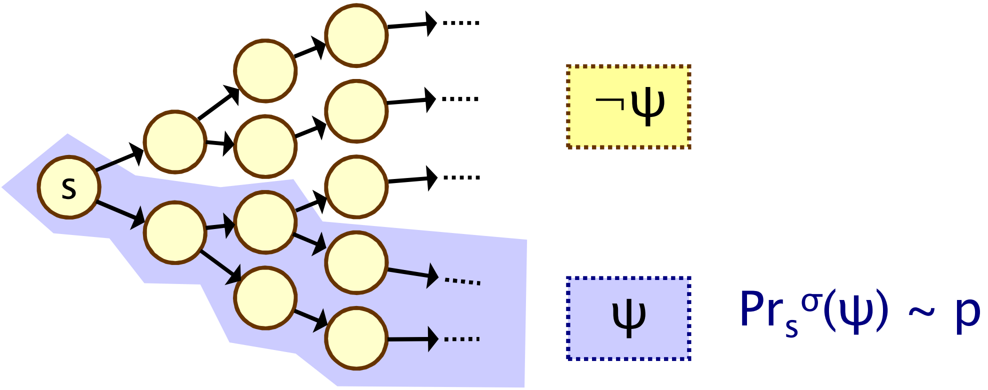
- PCTL formulas interpreted over states of an MDP
  - $s \vDash \phi$ denotes ϕ is "true in state s" or "satisfied in state s"

- Semantics of (non-probabilistic) state formulas:
  - for a state s of the MDP $(S,s_{init},\alpha,\delta,L)$:
  - $s \vDash a \quad \Leftrightarrow \quad a \in L(s)$
  - $s \vDash \phi_1 \wedge \phi_2 \quad \Leftrightarrow \quad s \vDash \phi_1$ and $s \vDash \phi_2$
  - $s \vDash \neg\phi \quad \Leftrightarrow \quad s \vDash \phi$ is false

- Semantics of path formulas:
  - for a path $\omega = s_0(a_0,\mu_0)s_1(a_1,\mu_1)s_2\ldots$ in the MDP:
  - $\omega \vDash X\,\phi \quad \Leftrightarrow \quad s_1 \vDash \phi$
  - $\omega \vDash \phi_1\ U^{\leq k}\ \phi_2 \quad \Leftrightarrow \quad \exists i \leq k$ such that $s_i \vDash \phi_2$ and $\forall j<i,\ s_j \vDash \phi_1$
  - $\omega \vDash \phi_1\ U\ \phi_2 \quad \Leftrightarrow \quad \exists k \geq 0$ such that $\omega \vDash \phi_1\ U^{\leq k}\ \phi_2$

PCTL semantics for MDPs

- Semantics of the probabilistic operator P
  - can only define probabilities for a specific strategy σ
  - s ⊨ $P_{\sim p}$ [ψ] means "the probability, from state s, that ψ is true for an outgoing path satisfies ~p for all strategies σ"
  - formally s ⊨ $P_{\sim p}$ [ψ] ⇔ $Pr_s^\sigma(\psi) \sim p$ for all strategies σ
  - where we use $Pr_s^\sigma(\psi)$ to denote $Pr_s^\sigma \{ \omega \in Path_s^\sigma \mid \omega \vDash \psi \}$



- Some equivalences:
  - F ϕ ≡ ◊ ϕ ≡ true U ϕ (eventually, "future")
  - G ϕ ≡ □ ϕ ≡ ¬(F ¬ϕ) (always, "globally")

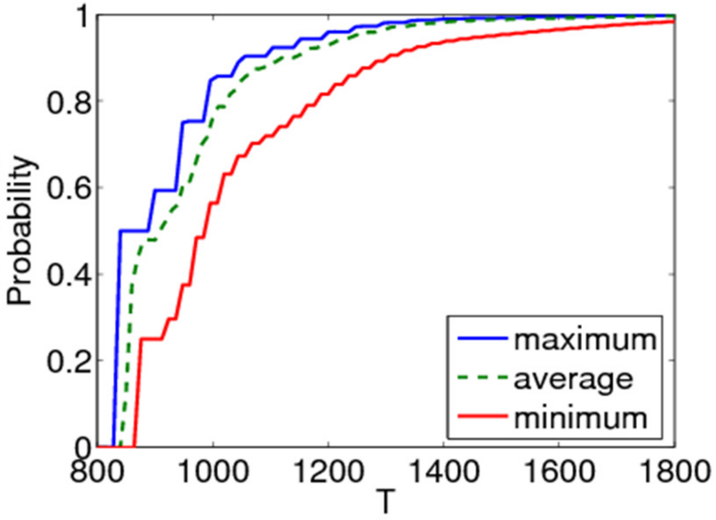
Minimum and maximum probabilities

- Letting:
  - $Pr_s^{max}(\psi) = \sup_\sigma Pr_s^\sigma(\psi)$
  - $Pr_s^{min}(\psi) = \inf_\sigma Pr_s^\sigma(\psi)$
- We have:
  - if $\sim \in \{\geq,>\}$, then $s \models P_{\sim p} [\ \psi\ ] \ \Leftrightarrow\ Pr_s^{min}(\psi) \sim p$
  - if $\sim \in \{<,\leq\}$, then $s \models P_{\sim p} [\ \psi\ ] \ \Leftrightarrow\ Pr_s^{max}(\psi) \sim p$
- Model checking $P_{\sim p}[\ \psi\ ]$ reduces to the computation over all strategies of either:
  - the minimum probability of ψ holding
  - the maximum probability of ψ holding
- Crucial result for model checking PCTL on MDPs
  - memoryless strategies suffice, i.e. there are always memoryless strategies σ_{min} and σ_{max} for which:
  - $Pr_s^{\sigma_{min}}(\psi) = Pr_s^{min}(\psi)$ and $Pr_s^{\sigma_{max}}(\psi) = Pr_s^{min}(\psi)$

Quantitative properties

- For PCTL properties with P as the outermost operator
  - quantitative form (two types): $P_{min=?}$ [ψ] and $P_{max=?}$ [ψ]
  - i.e. "what is the minimum/maximum probability (over all adversaries) that path formula ψ is true?"
  - corresponds to an analysis of best-case or worst-case behaviour of the system
  - model checking is no harder since compute the values of $Pr_s^{min}(\psi)$ or $Pr_s^{max}(\psi)$ anyway
  - useful to spot patterns/trends

- Example: CSMA/CD protocol
  - "min/max probability that a message is sent within the deadline"



Some real PCTL examples

- Byzantine agreement protocol
  - $P_{min=?}$ [F (agreement $\wedge$ rounds$\leq$2)]
  - “what is the minimum probability that agreement is reached within two rounds?”

- CSMA/CD communication protocol
  - $P_{max=?}$ [F collisions=k]
  - “what is the maximum probability of k collisions?”

- Self-stabilisation protocols
  - $P_{min=?}$ [$F^{\leq t}$ stable]
  - “what is the minimum probability of reaching a stable state within k steps?”

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PCTL model checking for MDPs

- Algorithm for PCTL model checking [BdA95]
  - inputs: MDP $M=(S,s_{init},\alpha,\delta,L)$, PCTL formula ϕ
  - output: $Sat(\phi) = \{ s \in S \mid s \models \phi \}$ = set of states satisfying ϕ

- Basic algorithm same as PCTL model checking for DTMCs
  - proceeds by induction on parse tree of ϕ
  - non-probabilistic operators (true, a, $\neg$, $\wedge$) straightforward

- Only need to consider $P_{\sim p} [\psi]$ formulas
  - reduces to computation of $Pr_s^{min}(\psi)$ or $Pr_s^{max}(\psi)$ for all $s \in S$
  - dependent on whether $\sim \in \{\geq,>\}$ or $\sim \in \{<,\leq\}$
  - these slides cover the case $Pr_s^{min}(\phi_1 \text{ U } \phi_2)$, i.e. $\sim \in \{\geq,>\}$
  - case for maximum probabilities is very similar
  - next ($X\ \phi$) and bounded until ($\phi_1 \text{ U}^{\leq k} \phi_2$) are straightforward extensions of the DTMC case

PCTL until for MDPs

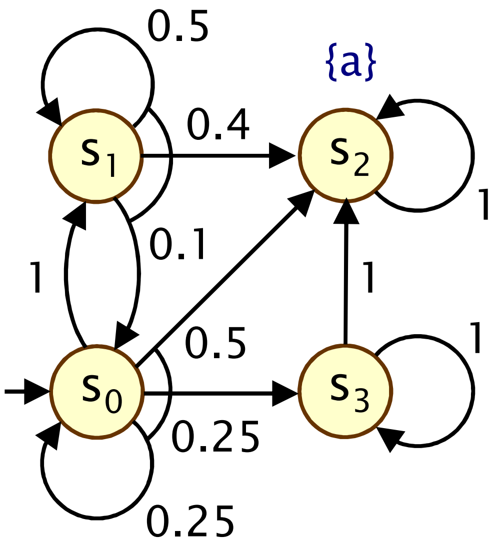
- Computation of probabilities $Pr_s^{min}(\phi_1\ U\ \phi_2)$ for all $s \in S$
- First identify all states where the probability is 1 or 0
  - "precomputation" algorithms, yielding sets S^{yes}, S^{no}
- Then compute (min) probabilities for remaining states ($S^?$)
  - either: solve linear programming problem
  - or: approximate with an iterative solution method
  - or: use policy iteration

Example:

$P_{\geq p}$ [F a]

$\equiv$

$P_{\geq p}$ [true U a]

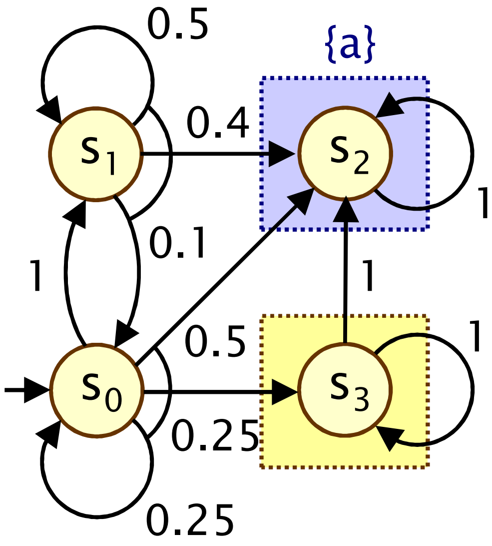


PCTL until – Precomputation

- Identify all states where $Pr_s^{min}(\phi_1 \text{ U } \phi_2)$ is 1 or 0
  - S^{yes} = $Sat(P_{\geq 1} [\ \phi_1 \text{ U } \phi_2\ ])$, S^{no} = $Sat(\neg\ P_{>0} [\ \phi_1 \text{ U } \phi_2\ ])$
- Two graph-based precomputation algorithms:
  - algorithm Prob1A computes S^{yes}
    - for all strategies the probability of satisfying ϕ_1 U ϕ_2 is 1
  - algorithm Prob0E computes S^{no}
    - there exists a strategy for which the probability is 0

Example:

$P_{\geq p}$ [F a]



S^{yes} = $Sat(P_{\geq 1}$ [F a])

S^{no} = $Sat(\neg P_{>0}$ [F a])

Method 1 – Linear programming

- Probabilities $Pr_s^{min}(\phi_1 \text{ U } \phi_2)$ for remaining states in the set $S^? = S \setminus (S^{yes} \cup S^{no})$ can be obtained as the unique solution of the following linear programming (LP) problem:

$$\text{maximize } \sum_{s \in S^?} x_s \text{ subject to the constraints :}$$

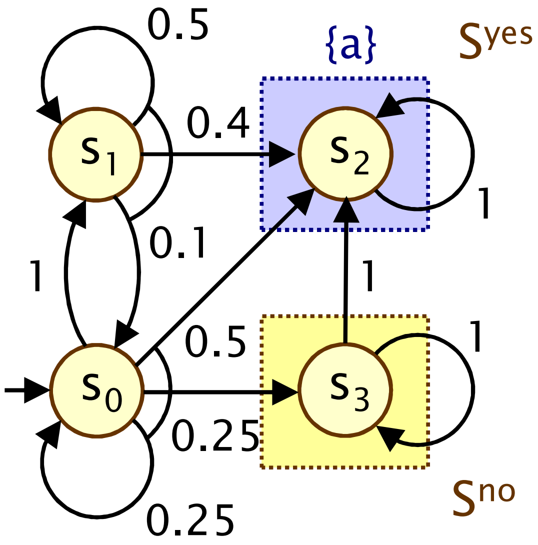
$$x_s \leq \sum_{s' \in S^?} \mu(s') \cdot x_{s'} + \sum_{s' \in S^{yes}} \mu(s')$$

$$\text{for all } s \in S^? \text{ and for all } (a, \mu) \in \delta(s)$$

- Simple case of a more general problem known as the stochastic shortest path problem [BT91]

- This can be solved with standard techniques
  - e.g. Simplex, ellipsoid method, branch-and-cut

Example – PCTL until (LP)



Let $x_i = Pr_{s_i}^{min}(F\ a)$

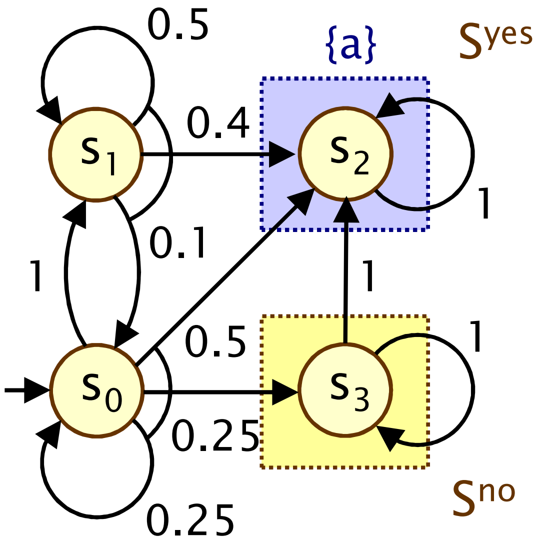
S^{yes}: $x_2=1$, S^{no}: $x_3=0$

For $S^? = \{x_0, x_1\}$:

Maximise x_0+x_1 subject to constraints:

- $x_0 \leq x_1$
- $x_0 \leq 0.25 \cdot x_0 + 0.5$
- $x_1 \leq 0.1 \cdot x_0 + 0.5 \cdot x_1 + 0.4$

Example – PCTL until (LP)



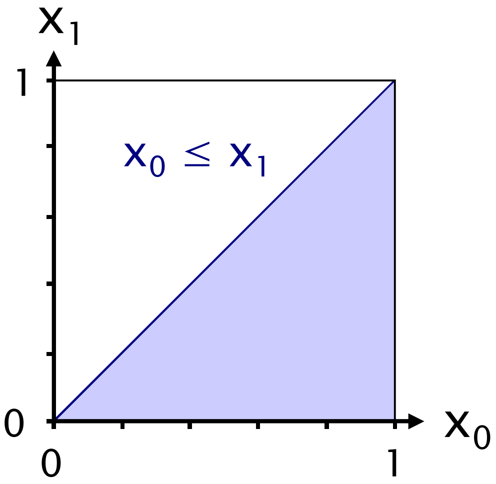
Let $x_i = Pr_{s_i}^{min}(F\ a)$

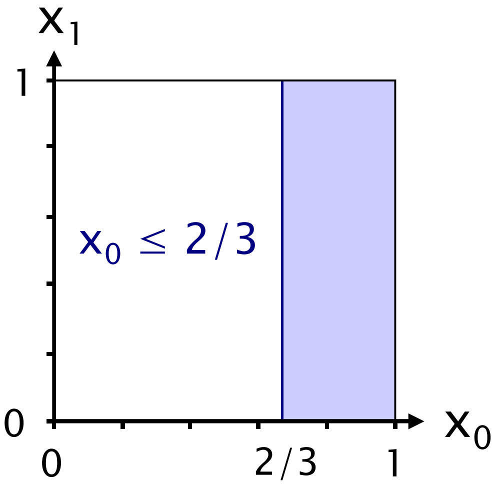
S^{yes}: $x_2=1$, S^{no}: $x_3=0$

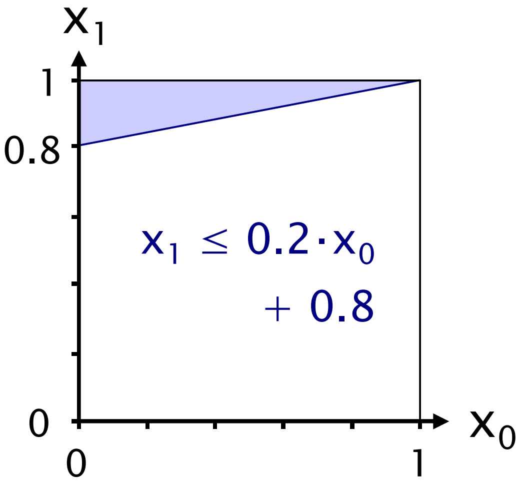
For $S^? = \{x_0, x_1\}$:

Maximise x_0+x_1 subject to constraints:

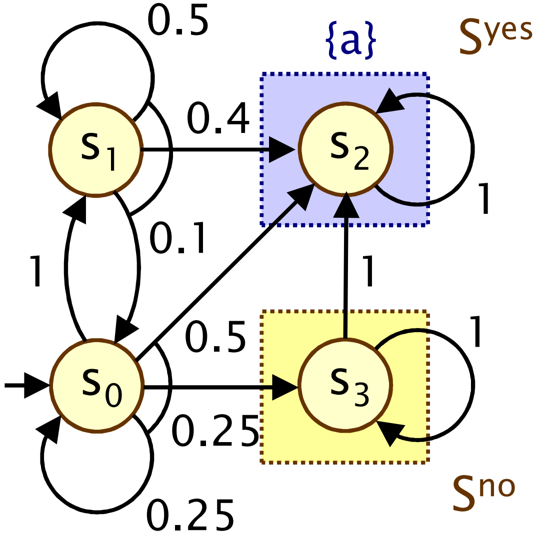
- $x_0 \leq x_1$
- $x_0 \leq 2/3$
- $x_1 \leq 0.2 \cdot x_0 + 0.8$







Example – PCTL until (LP)



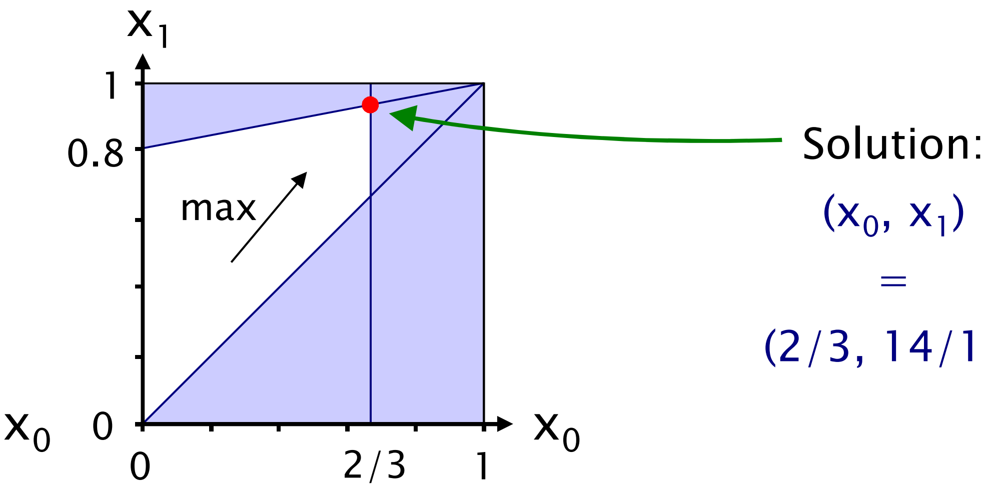
Let $x_i = Pr_{s_i}^{min}(F\ a)$

S^{yes}: $x_2=1$, S^{no}: $x_3=0$

For $S^? = \{x_0, x_1\}$:

Maximise x_0+x_1 subject to constraints:

- $x_0 \leq x_1$
- $x_0 \leq 2/3$
- $x_1 \leq 0.2 \cdot x_0 + 0.8$



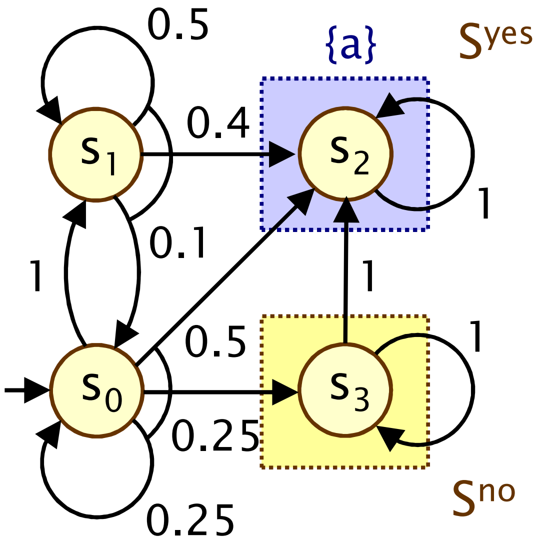
Solution:

(x_0, x_1)

$=$

$(2/3, 14/15)$

Example – PCTL until (LP)



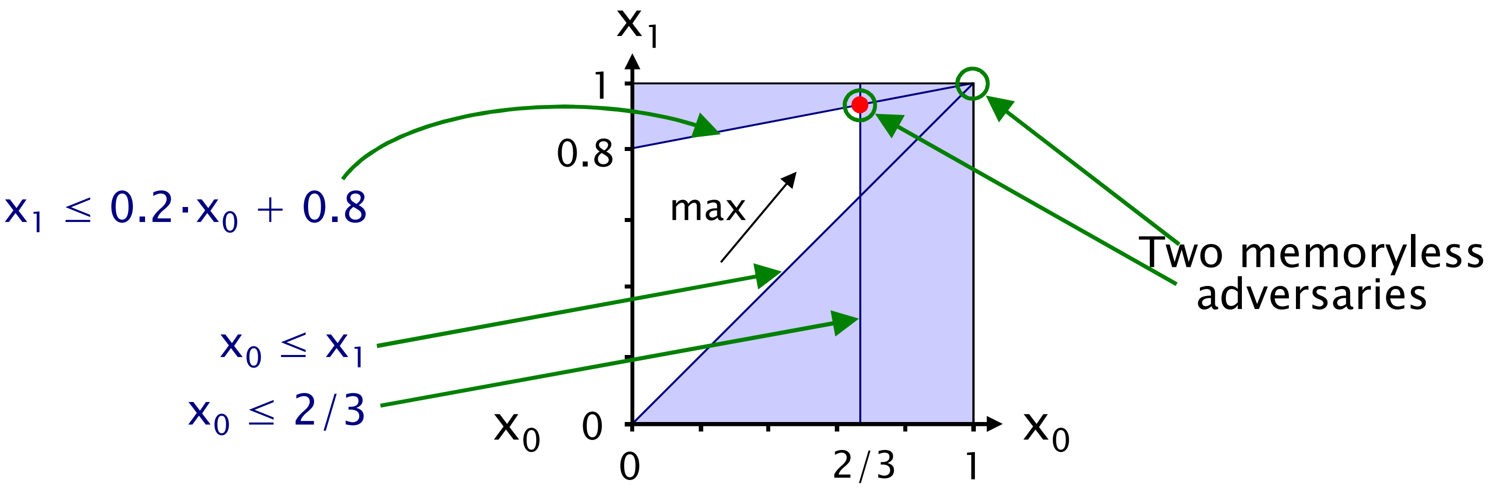
Let $x_i = Pr_{s_i}^{min}(F\ a)$

S^{yes}: $x_2=1$, S^{no}: $x_3=0$

For $S^? = \{x_0, x_1\}$:

Maximise x_0+x_1 subject to constraints:

- $x_0 \leq x_1$
- $x_0 \leq 2/3$
- $x_1 \leq 0.2 \cdot x_0 + 0.8$



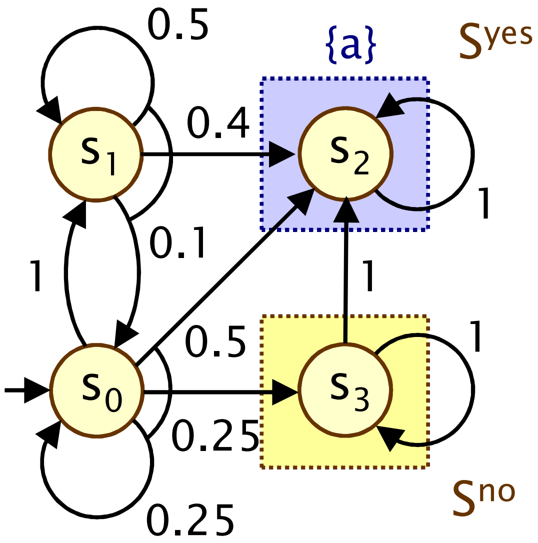
Method 2 – Value iteration

- For probabilities $Pr_s^{min}(\phi_1 \text{ U } \phi_2)$ it can be shown that:
  - $Pr_s^{min}(\phi_1 \text{ U } \phi_2) = \lim_{n \to \infty} x_s^{(n)}$ where:

$$x_s^{(n)} = \begin{cases} 1 & \text{if } s \in S^{yes} \\ 0 & \text{if } s \in S^{no} \\ 0 & \text{if } s \in S^{?} \text{ and } n = 0 \\ \min_{(a,\mu) \in Steps(s)} \left(\sum_{s' \in S} \mu(s') \cdot x_{s'}^{(n-1)} \right) & \text{if } s \in S^{?} \text{ and } n > 0 \end{cases}$$

- This forms the basis for an (approximate) iterative solution
  - iterations terminated when solution converges sufficiently

Example – PCTL until (value iteration)



Compute: $Pr_{s_i}^{min}(F\ a)$

$S^{yes} = \{x_2\}$, $S^{no} = \{x_3\}$, $S^? = \{x_0, x_1\}$

$[\ x_0^{(n)}, x_1^{(n)}, x_2^{(n)}, x_3^{(n)}\ ]$

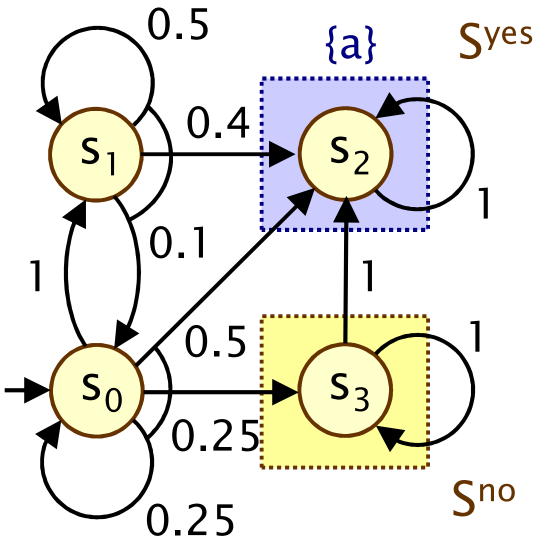
n=0: $[\ 0, 0, 1, 0\ ]$

n=1: $[\ \min(0, 0.25 \cdot 0 + 0.5),$
$0.1 \cdot 0 + 0.5 \cdot 0 + 0.4, 1, 0\ ]$
$= [\ 0, 0.4, 1, 0\ ]$

n=2: $[\ \min(0.4, 0.25 \cdot 0 + 0.5),$
$0.1 \cdot 0 + 0.5 \cdot 0.4 + 0.4, 1, 0\ ]$
$= [\ 0.4, 0.6, 1, 0\ ]$

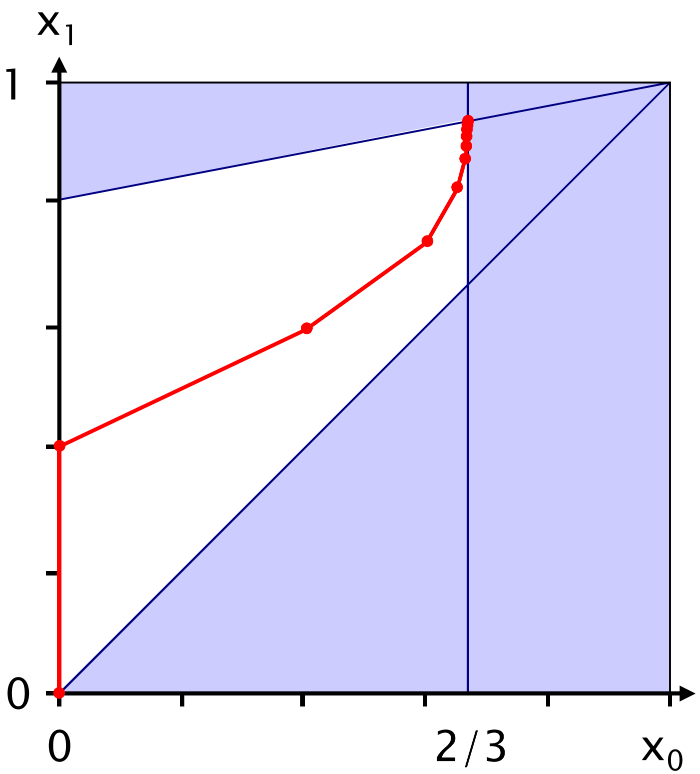
n=3: …

Example – PCTL until (value iteration)



	$[\, x_0^{(n)}, x_1^{(n)}, x_2^{(n)}, x_3^{(n)} \,]$
n=0:	[0.000000, 0.000000, 1, 0]
n=1:	[0.000000, 0.400000, 1, 0]
n=2:	[0.400000, 0.600000, 1, 0]
n=3:	[0.600000, 0.740000, 1, 0]
n=4:	[0.650000, 0.830000, 1, 0]
n=5:	[0.662500, 0.880000, 1, 0]
n=6:	[0.665625, 0.906250, 1, 0]
n=7:	[0.666406, 0.919688, 1, 0]
n=8:	[0.666602, 0.926484, 1, 0]
n=9:	[0.666650, 0.929902, 1, 0]
	…
n=20:	[0.666667, 0.933332, 1, 0]
n=21:	[0.666667, 0.933332, 1, 0]
	≈ [2/3, 14/15, 1, 0]

Example – Value iteration + LP



$[\ x_0^{(n)}, x_1^{(n)}, x_2^{(n)}, x_3^{(n)}\ ]$

n=0: [0.000000, 0.000000, 1, 0]

n=1: [0.000000, 0.400000, 1, 0]

n=2: [0.400000, 0.600000, 1, 0]

n=3: [0.600000, 0.740000, 1, 0]

n=4: [0.650000, 0.830000, 1, 0]

n=5: [0.662500, 0.880000, 1, 0]

n=6: [0.665625, 0.906250, 1, 0]

n=7: [0.666406, 0.919688, 1, 0]

n=8: [0.666602, 0.926484, 1, 0]

n=9: [0.666650, 0.929902, 1, 0]

…

n=20: [0.666667, 0.933332, 1, 0]

n=21: [0.666667, 0.933332, 1, 0]

$\approx$ [2/3, 14/15, 1, 0]

Method 3 – Policy iteration

- Value iteration:
  - iterates over (vectors of) probabilities
- Policy iteration:
  - iterates over strategies (“policies”)

- 1. Start with an arbitrary (memoryless) strategy σ
- 2. Compute the reachability probabilities $\underline{Pr}^{\sigma}(F\ a)$ for σ
- 3. Improve the strategy in each state
- 4. Repeat 2/3 until no change in strategy

- Termination:
  - finite number of memoryless strategies
  - improvement in (minimum) probabilities each time

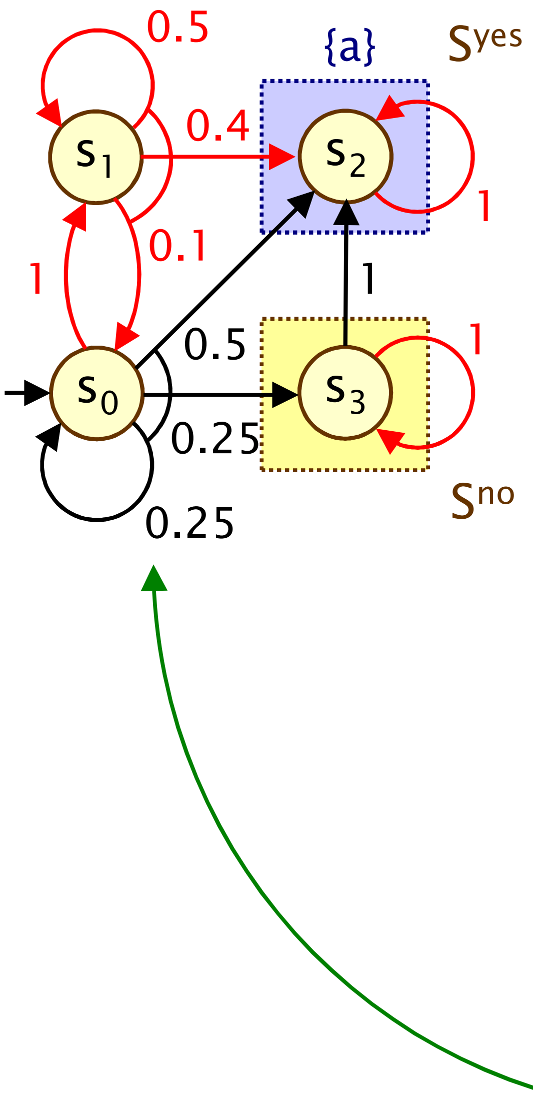
Method 3 – Policy iteration

- 1. Start with an arbitrary (memoryless) strategy σ
  - pick an element of $\delta(s)$ for each state $s \in S$
- 2. Compute the reachability probabilities $\underline{Pr}^{\sigma}(F\ a)$ for σ
  - probabilistic reachability on a DTMC
  - i.e. solve linear equation system
- 3. Improve the strategy in each state

$$\sigma'(s) = \operatorname{argmin} \left\{ \sum_{s' \in S} \mu(s') \cdot Pr^{\sigma}_{s'}(F\ a) \mid (a, \mu) \in \delta(s) \right\}$$

- 4. Repeat 2/3 until no change in strategy

Example – Policy iteration



Arbitrary strategy σ:

Compute: $\underline{Pr}^{\sigma}(F\ a)$

Let $x_i = Pr_{s_i}{}^{\sigma}(F\ a)$

$x_2=1$, $x_3=0$ and:

- $x_0 = x_1$
- $x_1 = 0.1 \cdot x_0 + 0.5 \cdot x_1 + 0.4$

Solution:

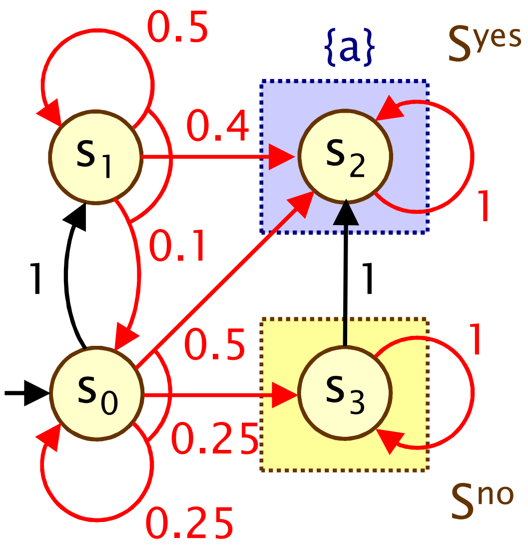
$\underline{Pr}^{\sigma}(F\ a) = [\ 1, 1, 1, 0\ ]$

Refine σ in state s_0:

$\min\{1(1), 0.5(1)+0.25(0)+0.25(1)\}$

$= \min\{1, 0.75\} = 0.75$

Example – Policy iteration



Refined strategy σ':

Compute: $\underline{\Pr}^{\sigma'}(F\ a)$

Let $x_i = \Pr_{s_i}^{\sigma'}(F\ a)$

$x_2=1$, $x_3=0$ and:

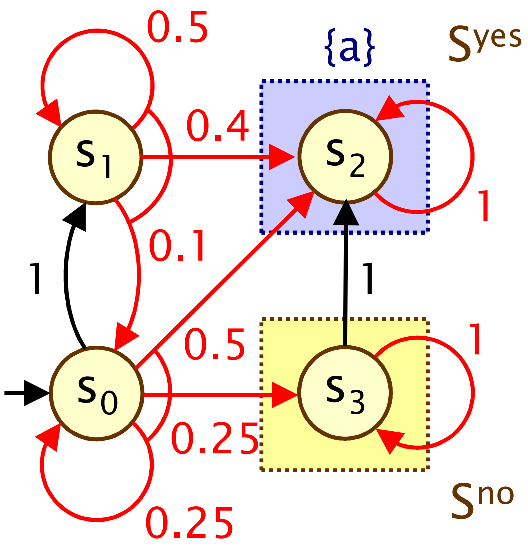
- $x_0 = 0.25 \cdot x_0 + 0.5$
- $x_1 = 0.1 \cdot x_0 + 0.5 \cdot x_1 + 0.4$

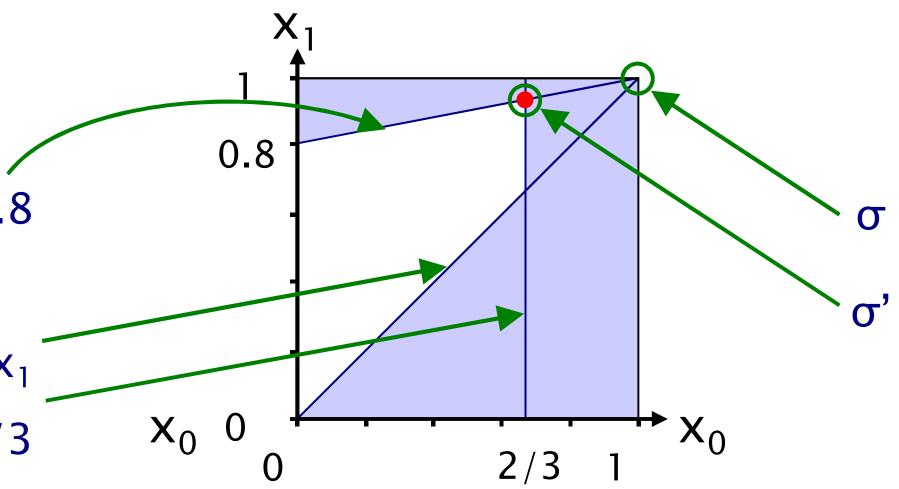
Solution:

$\underline{\Pr}^{\sigma'}(F\ a) = [\ 2/3,\ 14/15,\ 1,\ 0\ ]$

This is optimal

Example – Policy iteration





$x_1 = 0.2 \cdot x_0 + 0.8$

$x_0 = x_1$

$x_0 = 2/3$

σ

σ'

PCTL model checking – Summary

- Computation of set Sat(Φ) for MDP M and PCTL formula Φ
  - recursive descent of parse tree
  - combination of graph algorithms, numerical computation

- Probabilistic operator P:
  - X Φ : one matrix-vector multiplication, $O(|S|^2)$
  - Φ_1 $U^{\leq k}$ Φ_2 : k matrix-vector multiplications, $O(k|S|^2)$
  - Φ_1 U Φ_2 : linear programming problem, polynomial in |S| (assuming use of linear programming)

- Complexity:
  - linear in |Φ| and polynomial in |S|
  - S is states in MDP, assume |δ(s)| is constant

Costs and rewards for MDPs

- We can augment MDPs with rewards (or, conversely, costs)
  - real-valued quantities assigned to states and/or transitions
  - these can have a wide range of possible interpretations
- Some examples:
  - elapsed time, power consumption, size of message queue, number of messages successfully delivered, net profit

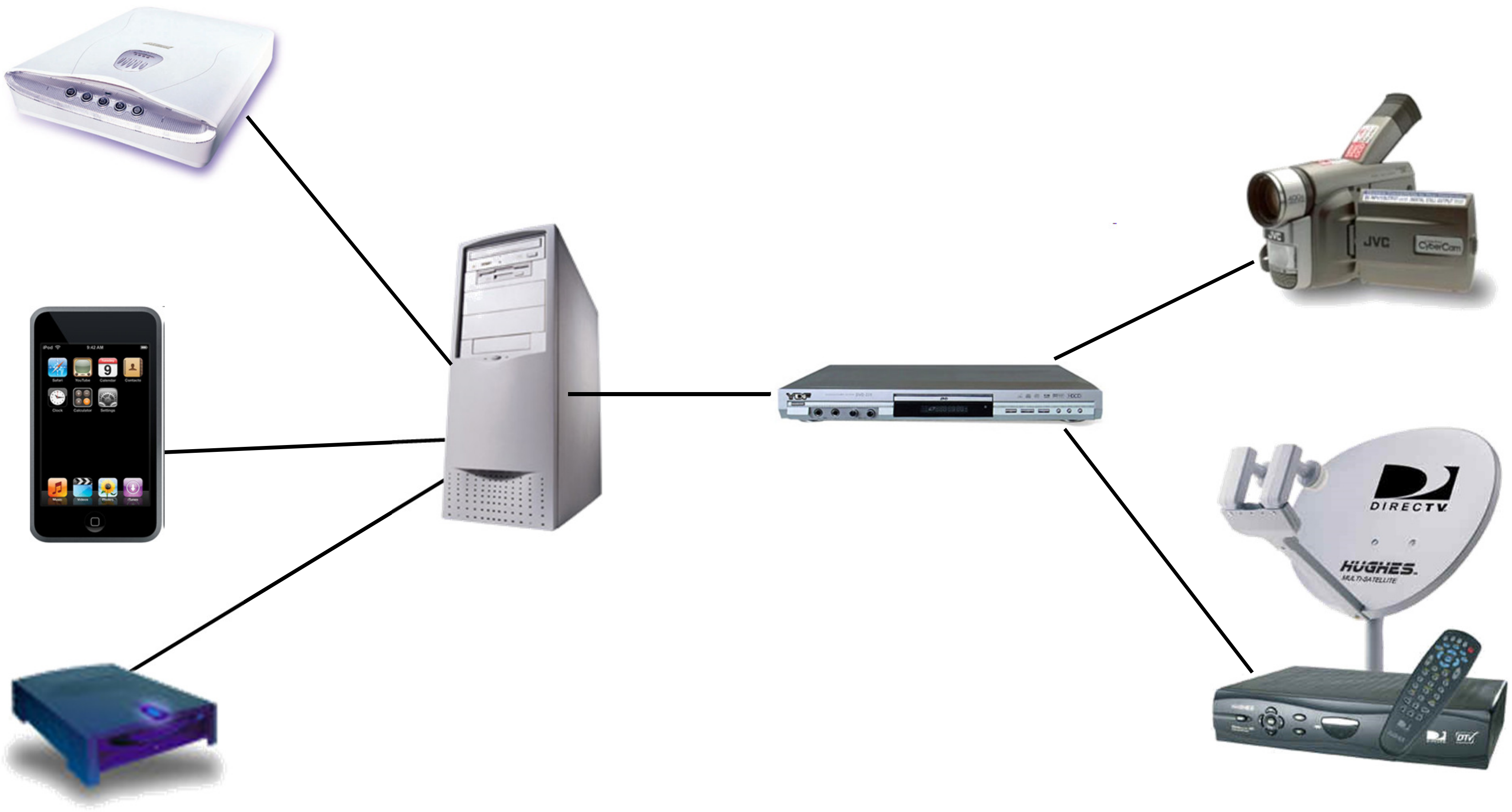
- Extend logic PCTL with R operator, for "expected reward"
  - as for PCTL, either $R_{\sim r}$ […], $R_{min=?}$ […] or $R_{max=?}$ […]
- Some examples:
  - $R_{min=?}$ [$I^{=90}$], $R_{max=?}$ [$C^{\leq 60}$], $R_{max=?}$ [F "end"]
  - "the minimum expected queue size after exactly 90 seconds"
  - "the maximum expected power consumption over one hour"
  - the maximum expected time for the algorithm to terminate

Case study: FireWire root contention

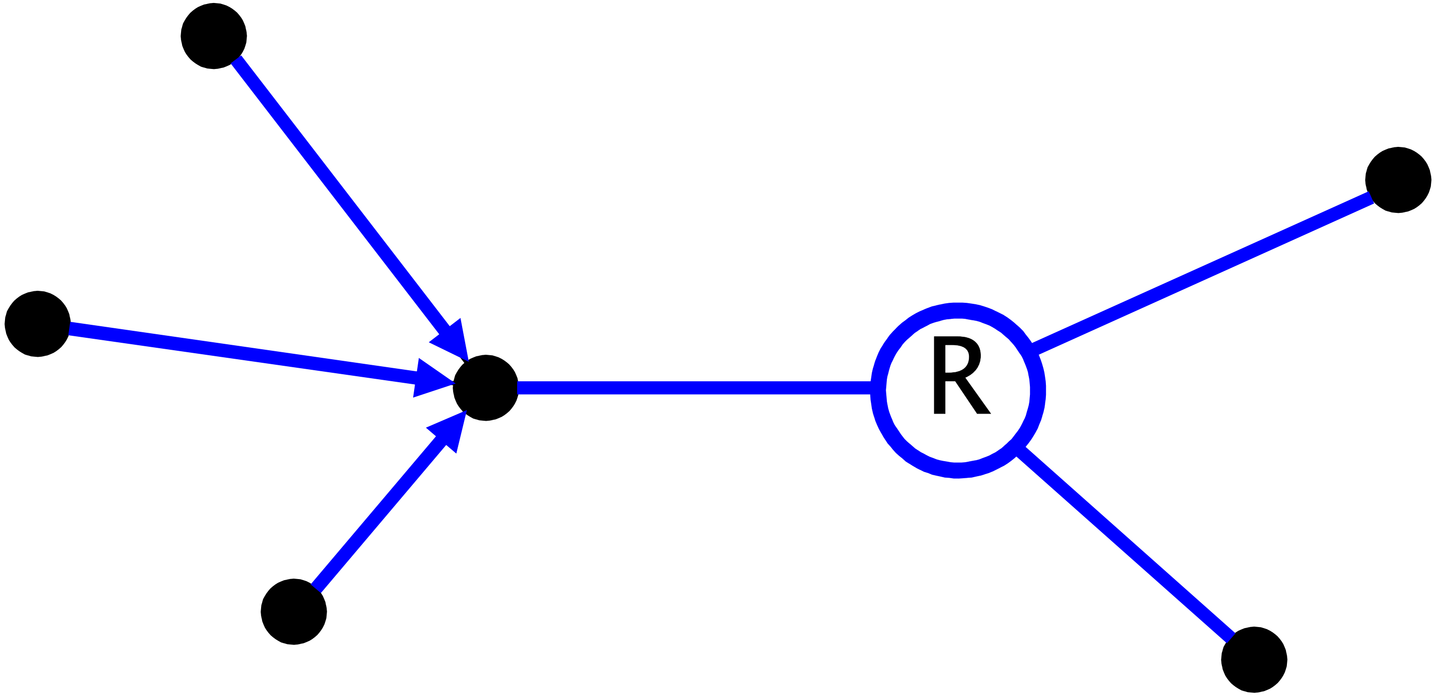
- FireWire (IEEE 1394)
  - high-performance serial bus for networking multimedia devices; originally by Apple
  - "hot-pluggable" - add/remove devices at any time
  - no requirement for a single PC (but need acyclic topology)

- Root contention protocol
  - leader election algorithm, when nodes join/leave
  - symmetric, distributed protocol
  - uses randomisation (electronic coin tossing) and timing delays
  - nodes send messages: "be my parent"
  - root contention: when nodes contend leadership
  - random choice: "fast"/"slow" delay before retry

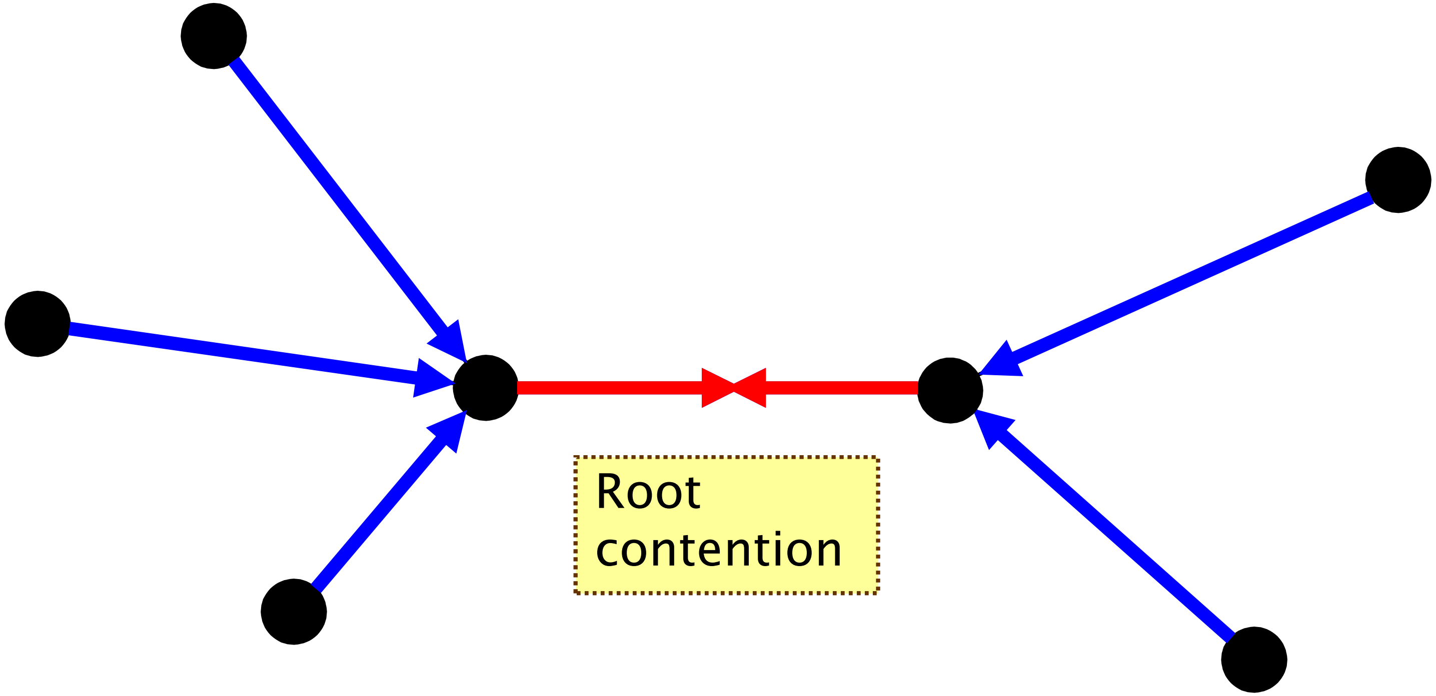
FireWire example



FireWire leader election

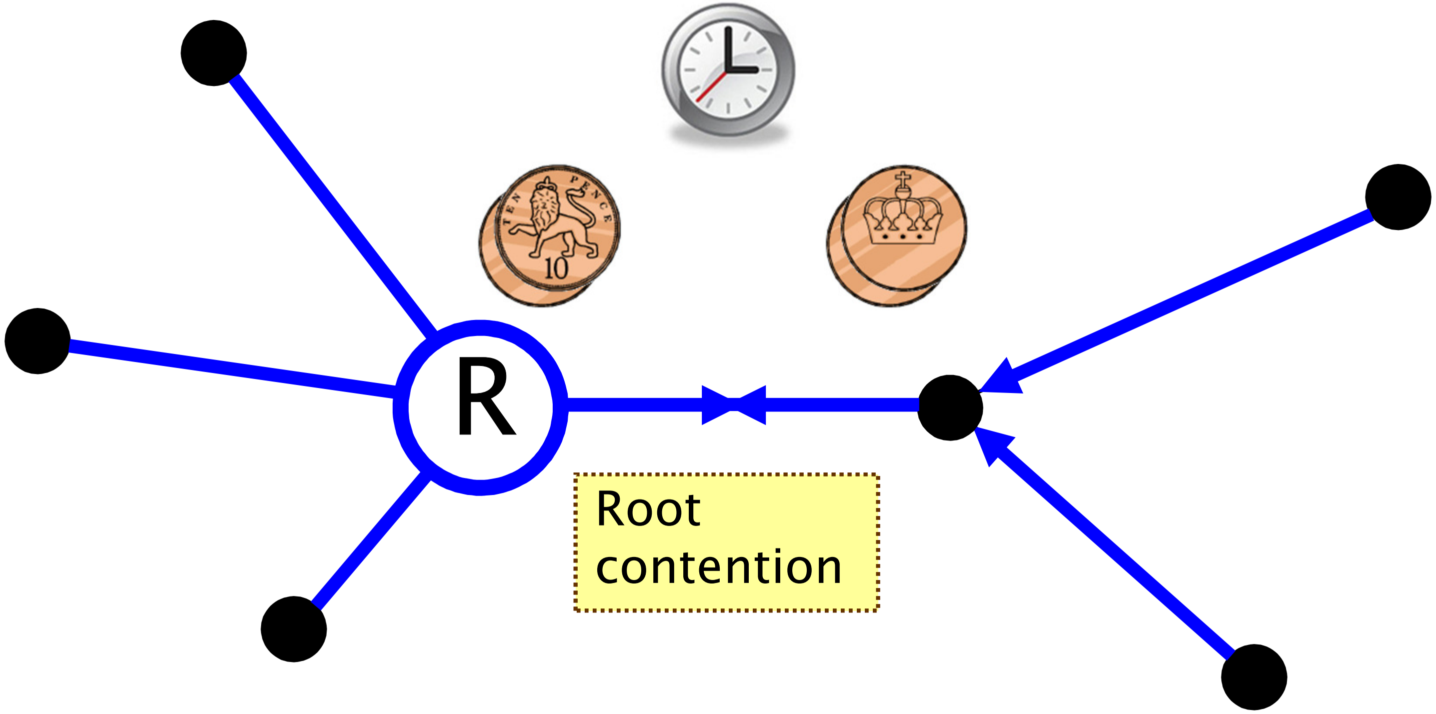


FireWire root contention



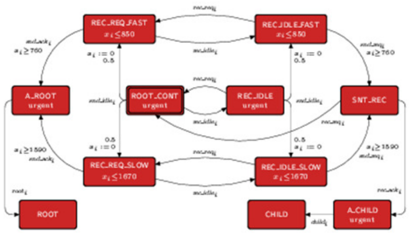
Root contention

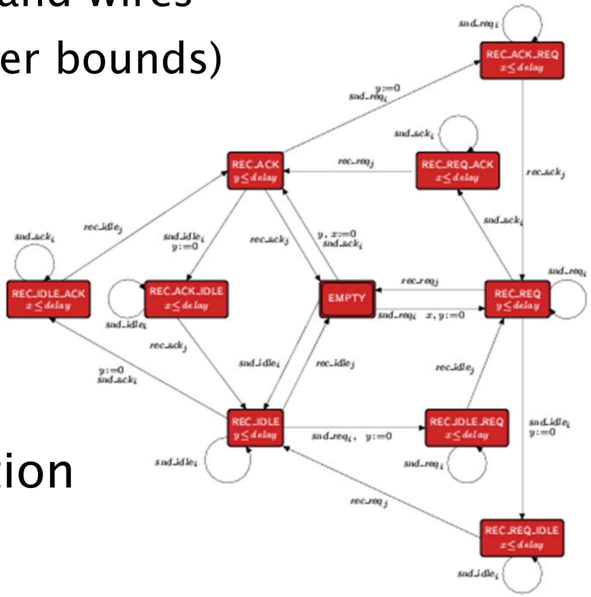
FireWire root contention



FireWire analysis

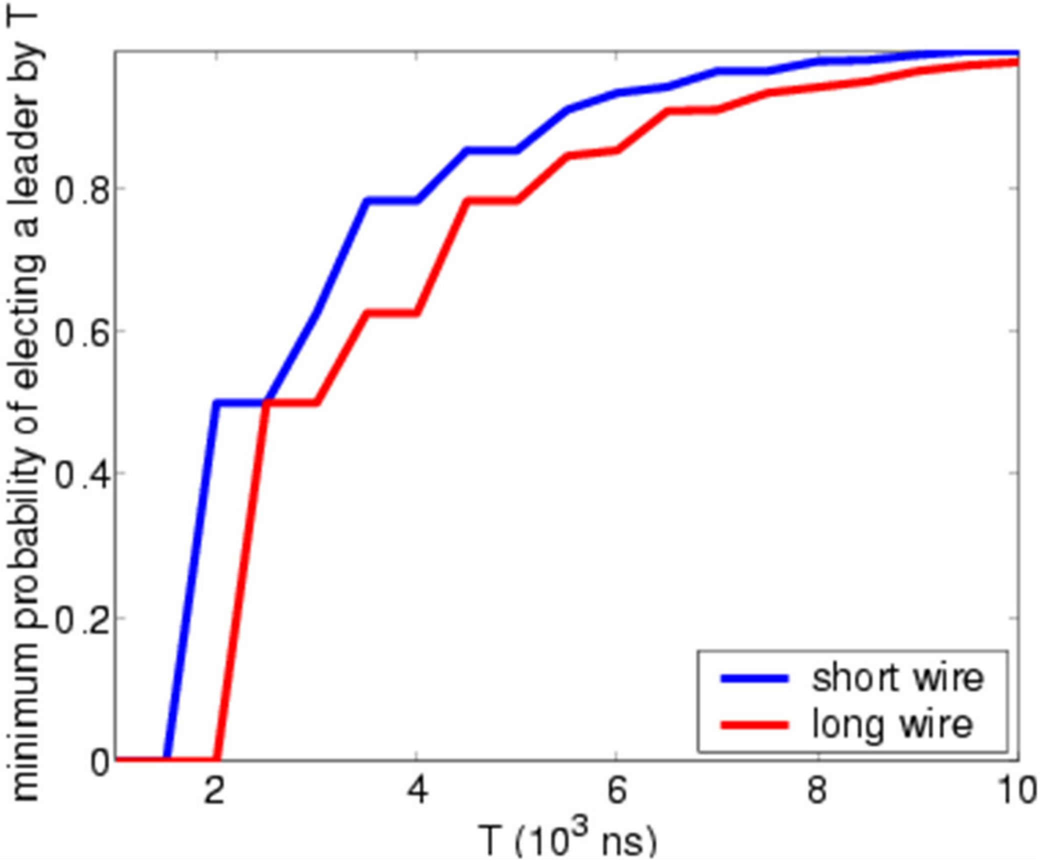
- Probabilistic model checking
  - model constructed and analysed using PRISM
  - timing delays taken from IEEE standard
  - model includes:
    - concurrency: messages between nodes and wires
    - underspecification of delays (upper/lower bounds)
  - max. model size: 170 million states





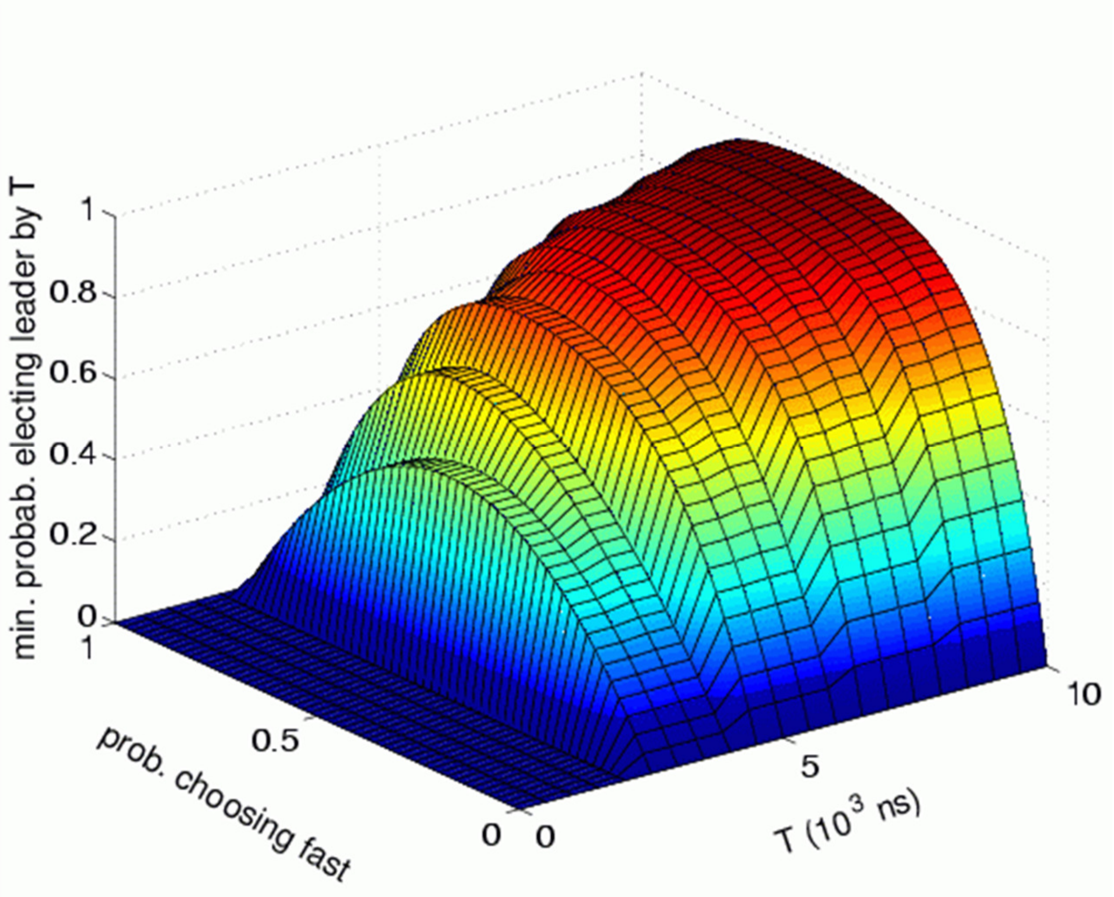
- Analysis:
  - verified that root contention always resolved with probability 1
  - investigated time taken for leader election
  - and the effect of using biased coin
    - based on a conjecture by Stoelinga

FireWire: Analysis results



"minimum probability of electing leader by time T"

FireWire: Analysis results

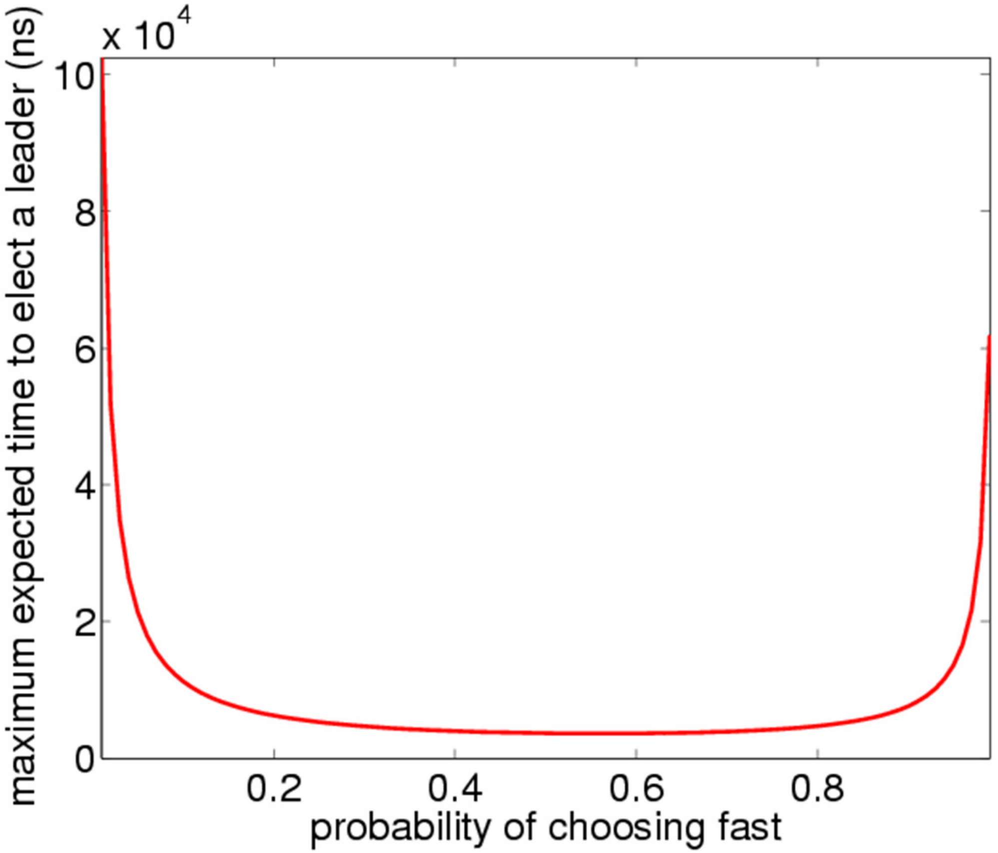


“minimum probability of electing leader by time T”

(short wire length)

Using a biased coin

FireWire: Analysis results

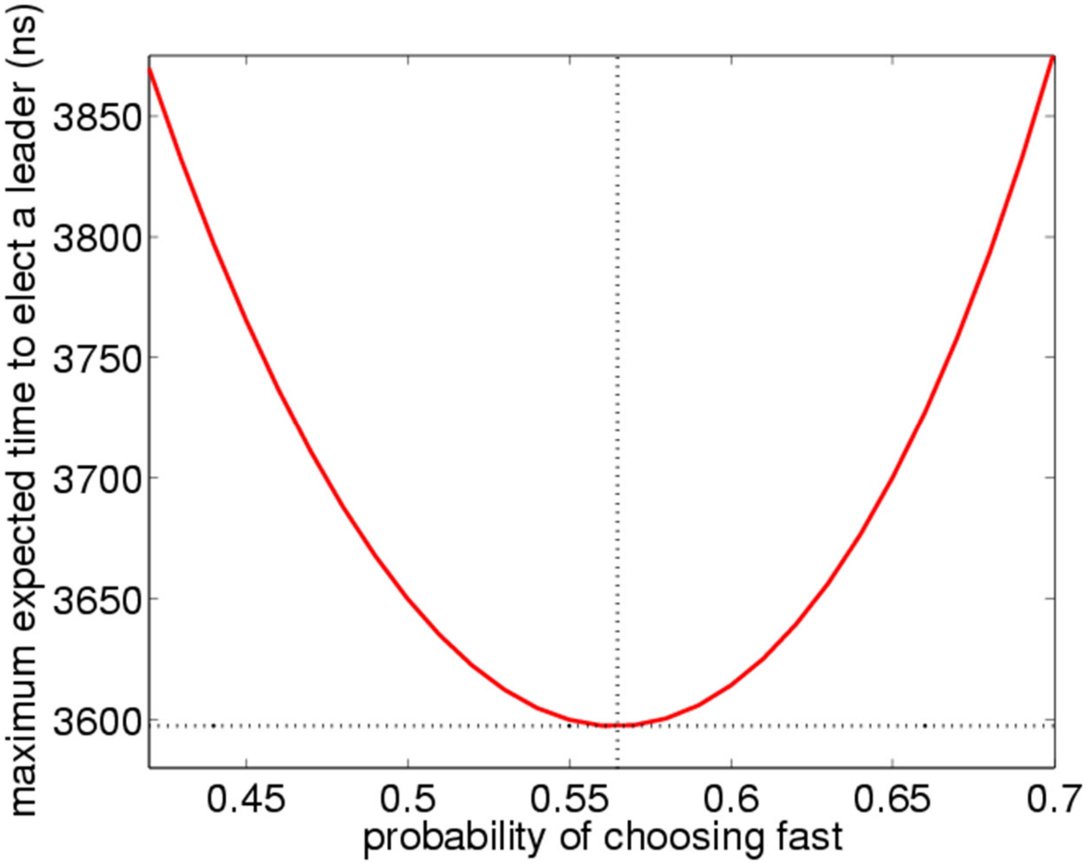


"maximum expected time to elect a leader"

(short wire length)

Using a biased coin

FireWire: Analysis results



“maximum expected time to elect a leader”

(short wire length)

Using a biased coin is beneficial!

Overview (Part 2)

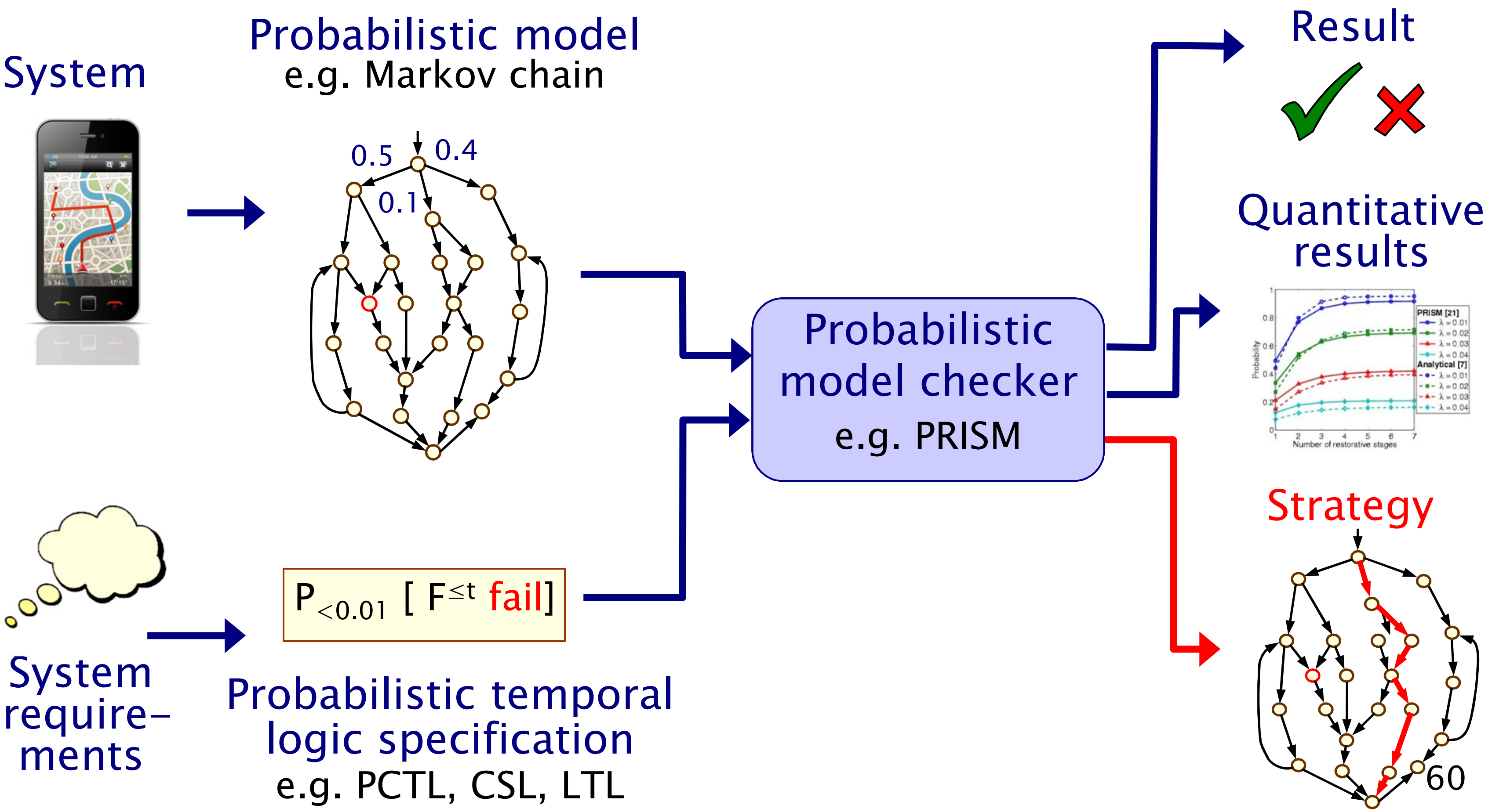
- Introduction
- Model checking for Markov decision processes (MDPs)
  - MDPs: definition
  - Paths, strategies & probability spaces
  - PCTL model checking
  - Costs and rewards
  - Case study: Firewire root contention
- Strategy synthesis for MDPs
  - Properties and objectives
  - Verification vs synthesis
  - Case study: Dynamic power management
- Summary

From verification to synthesis

- Shift towards quantitative model synthesis from speciication
  - begin with simpler problems: strategy synthesis, template-based synthesis, etc
  - advantage: correct-by-construction
- Here consider the problem of strategy (controller) synthesis
  - i.e. “can we construct a strategy to guarantee that a given quantitative property is satisfied?”
  - instead of “does the model satisfy a given quantitative property?”
  - also parameter synthesis: “find optimal value for parameter to satisfy quantitative objective”
- Many application domains
  - robotics (controller synthesis from LTL/PCTL)
  - dynamic power management (optimal policy synthesis)

Quantitative (probabilistic) verification

Automatic verification and strategy synthesis from quantitative properties for probabilistic models



System

Probabilistic model
e.g. Markov chain

System require-ments

$P_{<0.01}$ [$F^{\leq t}$ fail]

Probabilistic temporal logic specification
e.g. PCTL, CSL, LTL

Probabilistic model checker
e.g. PRISM

Result

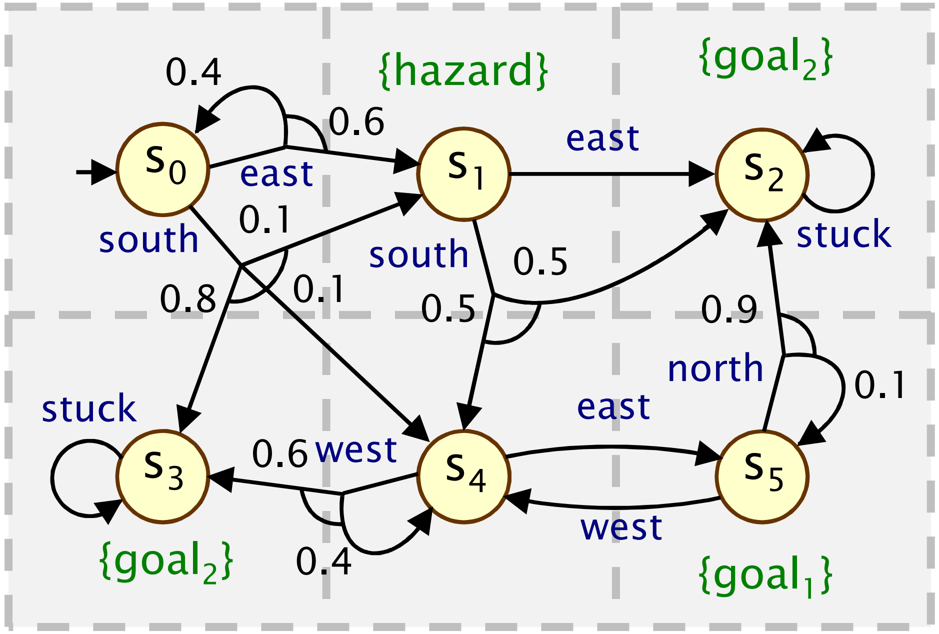
✓ ✗

Quantitative results

Strategy

Running example

- Example MDP
  - robot moving through terrain divided into 3 x 2 grid



States:

s_0, s_1, s_2, s_3, s_4, s_5

Actions:

north, east, south,
west, stuck

Labels

(atomic propositions):

hazard, goal$_1$, goal$_2$

Properties and objectives

- The syntax:

ψ is true with probability ~p

expected reward is ~r

  - ϕ ::= $P_{\sim p}$ [ψ] | $R_{\sim r}$ [ρ]
  - ψ ::= true | a | ψ ∧ ψ | ¬ ψ | X ψ | $\psi\ U^{\leq k}\ \psi$ | ψ U ψ

"next" "bounded until" "until"

  - ρ ::= F b | C | $C^{\leq k}$

"reachability" "cumulative"

  - where b is an atomic proposition, used to identify states of interest, p ∈ [0,1] is a probability, $\sim\ \in \{<,>,\leq,\geq\}$, $k \in \mathbb{N}$, and $r \in \mathbb{R}_{\geq 0}$
  - F b ≡ true U b
- We refer to ϕ as property, ψ and ρ as objectives
  - (branching time more challenging for synthesis)

Properties and objectives

- Semantics of the probabilistic operator P
  - can only define probabilities for a specific strategy σ
  - $s \vDash P_{\sim p} [\psi]$ means "the probability, from state s, that ψ is true for an outgoing path satisfies $\sim p$ for all strategies σ"
  - formally $s \vDash P_{\sim p} [\psi] \Leftrightarrow Pr_s^{\sigma}(\psi) \sim p$ for all strategies σ
  - where we use $Pr_s^{\sigma}(\psi)$ to denote $Pr_s^{\sigma} \{ \omega \in Path_s^{\sigma} \mid \omega \vDash \psi \}$

- $R_{\sim r} [\cdot]$ means "the expected value of $\cdot$ satisfies $\sim r$"

- Some examples:
  - $P_{\geq 0.4} [F \text{ "goal" }]$ "probability of reaching goal is at least 0.4"
  - $R_{<5} [C^{\leq 60}]$ "expected power consumption over one hour is below 5"
  - $R_{\leq 10} [F \text{ "end" }]$ "expected time to termination is at most 10"

Verification and strategy synthesis

- The verification problem is:
  - Given an MDP M and a property ϕ, does M satisfy ϕ under any possible strategy σ?

- The synthesis problem is dual:
  - Given an MDP M and a property ϕ, find, if it exists, a strategy σ such that M satisfies ϕ under σ

- Verification and strategy synthesis is achieved using <u>the same techniques</u>, namely computing optimal values for probability objectives:
  - $\mathrm{Pr}_s^{\min}(\psi) = \inf_\sigma \mathrm{Pr}_s^{\sigma}(\psi)$
  - $\mathrm{Pr}_s^{\max}(\psi) = \sup_\sigma \mathrm{Pr}_s^{\sigma}(\psi)$
  - and similarly for expectations

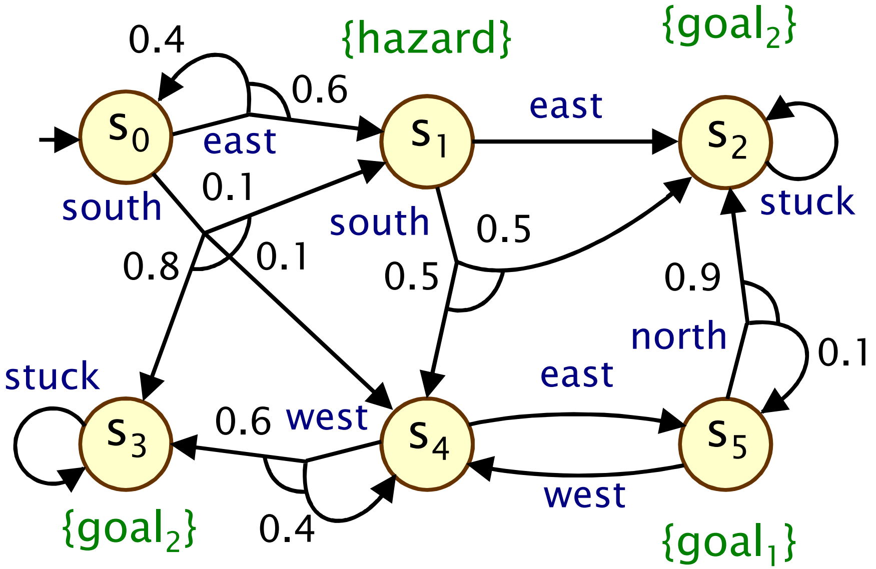
Computing reachability for MDPs

- Computation of probabilities $\mathrm{Pr}_s^{\max}(\mathrm{F}\ b)$ for all $s \in S$

- Step 1: pre-compute all states where probability is 1 or 0
  - graph-based algorithms, yielding sets S^{yes}, S^{no}
- Step 2: compute probabilities for remaining states ($S^?$)
  - (i) solve linear programming problem
  - (i) approximate with value iteration
  - (iii) solve with policy (strategy) iteration

- 1. Precomputation:
  - algorithm Prob1E computes S^{yes}
    - there exists a strategy for which the probability of "F b" is 1
  - algorithm Prob0A computes S^{no}
    - for all strategies, the probability of satisfying "F b" is 0

Example – Reachability



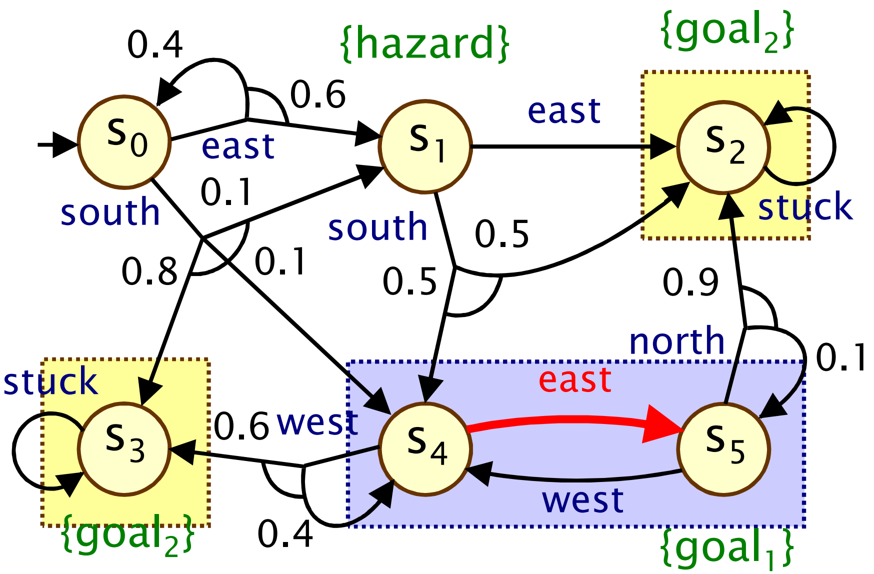
Example:

$P_{\geq 0.4}$ [F goal$_1$]

So compute:

Pr_s^{max}(F goal$_1$)

Example – Precomputation



S^{yes}

S^{no}

Example:

$P_{\geq 0.4}$ [F goal$_1$]

So compute:

$Pr_s^{max}(F\ goal_1)$

Reachability for MDPs

- 2. Numerical computation
  - compute probabilities $\mathrm{Pr}_s^{\max}(F\ b)$
  - for remaining states in $S^? = S \setminus (S^{yes} \cup S^{no})$
  - obtained as the unique solution of the linear programming (LP) problem:

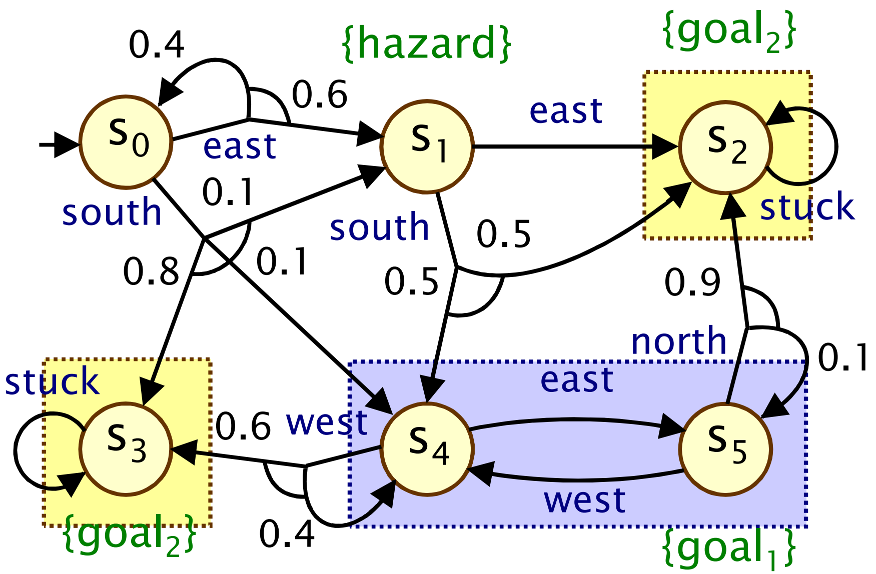
$$\text{minimize } \textstyle\sum_{s\in S^?} x_s \text{ subject to the constraints:}$$

$$x_s \geq \sum_{s'\in S^?} \delta(s,a)(s') \cdot x_{s'} + \sum_{s'\in S^{yes}} \delta(s,a)(s')$$

$$\text{for all } s \in S^? \text{ and for all } a \in A(s)$$

- This can be solved with standard techniques
  - e.g. Simplex, ellipsoid method, branch-and-cut

Example – Reachability (LP)



Example:

$P_{\geq 0.4}$ [F goal$_1$]

So compute:

Pr_s^{max}(F goal$_1$)

Let $x_i = Pr_{s_i}^{max}$(F goal$_1$)

S^{yes}: $x_4 = x_5 = 1$

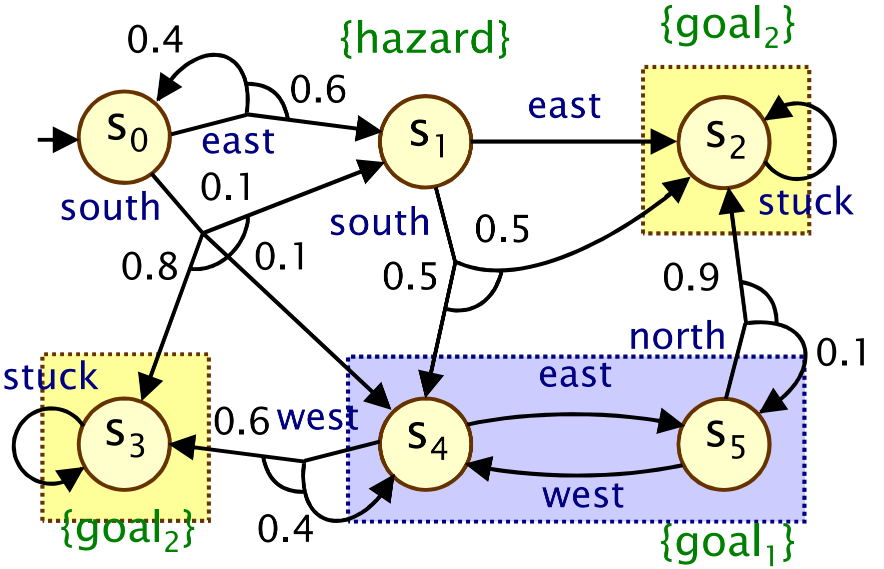
S^{no}: $x_2 = x_3 = 0$

For $S^? = \{x_0, x_1\}$:

Minimise $x_0 + x_1$ subject to:

- $x_0 \geq 0.4 \cdot x_0 + 0.6 \cdot x_1$ (east)
- $x_0 \geq 0.1 \cdot x_1 + 0.1$ (south)
- $x_1 \geq 0.5$ (south)
- $x_1 \geq 0$ (east)

Example – Reachability (LP)



Let $x_i = Pr_{s_i}^{max}(F\ goal_1)$

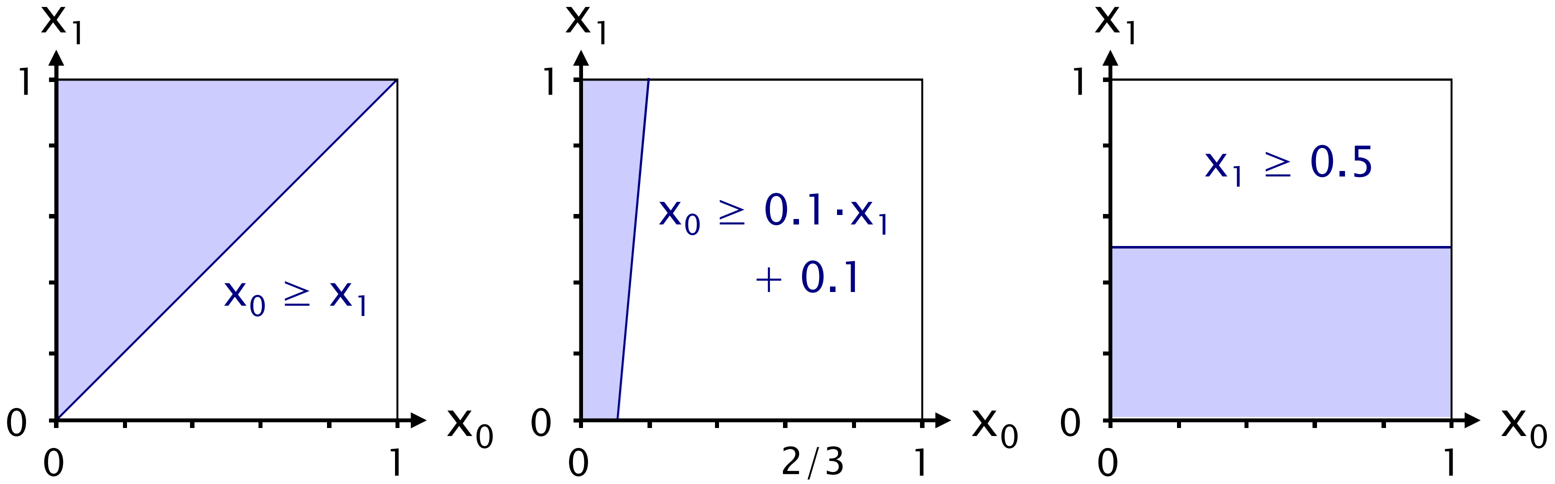
S^{yes}: $x_4=x_5=1$

S^{no}: $x_2=x_3=0$

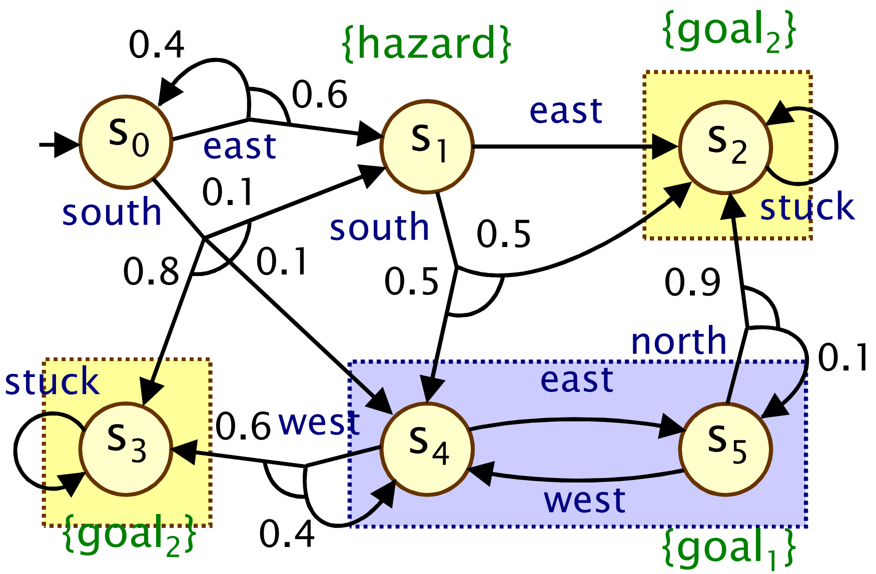
For $S^? = \{x_0, x_1\}$:

Minimise x_0+x_1 subject to:

- $x_0 \geq x_1$ (east)
- $x_0 \geq 0.1 \cdot x_1 + 0.1$ (south)
- $x_1 \geq 0.5$ (south)



Example – Reachability (LP)



Let $x_i = Pr_{s_i}^{max}(F\ goal_1)$

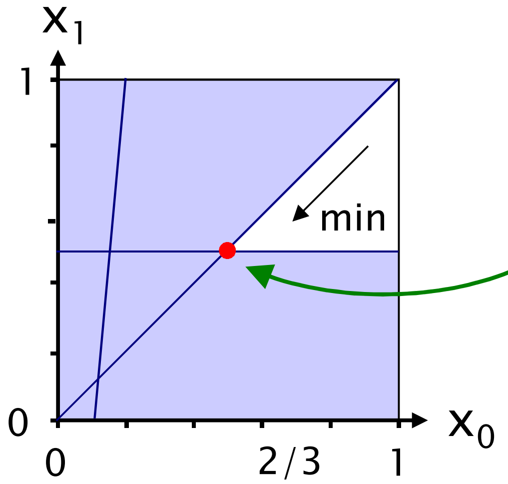
S^{yes}: $x_4 = x_5 = 1$

S^{no}: $x_2 = x_3 = 0$

For $S^? = \{x_0, x_1\}$:

Minimise $x_0 + x_1$ subject to:

- $x_0 \geq x_1$
- $x_0 \geq 0.1 \cdot x_1 + 0.1$
- $x_1 \geq 0.5$



Solution:

$(x_0, x_1) = (0.5, 0.5)$

i.e.

$Pr_{s_0}^{max}(F\ goal_1) = 0.5$

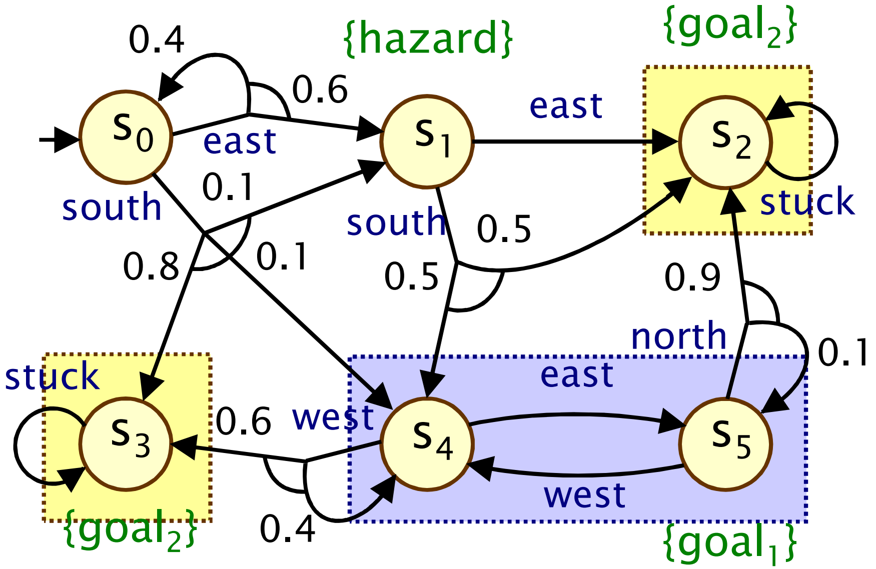
Reachability for MDPs

- 2. Numerical computation (alternative method)
  - value iteration
  - it can be shown that: $\mathrm{Pr}_s^{\max}(\mathrm{F}\ b) = \lim_{n\to\infty} x_s^{(n)}$ where:

$$x_s^{(n)} = \begin{cases} 1 & \text{if } s \in S^{yes} \\ 0 & \text{if } s \in S^{no} \\ 0 & \text{if } s \in S^{?} \text{ and } n = 0 \\ \max\left\{ \sum_{s' \in S} \delta(s,a)(s') \cdot x_{s'}^{(n-1)} \mid a \in A(s) \right\} & \text{if } s \in S^{?} \text{ and } n > 0 \end{cases}$$

- Approximate iterative solution technique
  - iterations terminated when solution converges sufficiently

Example - Reachability (val. iter.)



Compute: $Pr_s^{max}(F\ goal_1)$

S^{yes}: $x_4 = x_5 = 1$

S^{no}: $x_2 = x_3 = 0$

$S^? = \{x_0, x_1\}$

$[\ x_0^{(n)}, x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, x_4^{(n)}, x_5^{(n)}\ ]$

n=0: $[\ 0, 0, 0, 0, 1, 1\ ]$

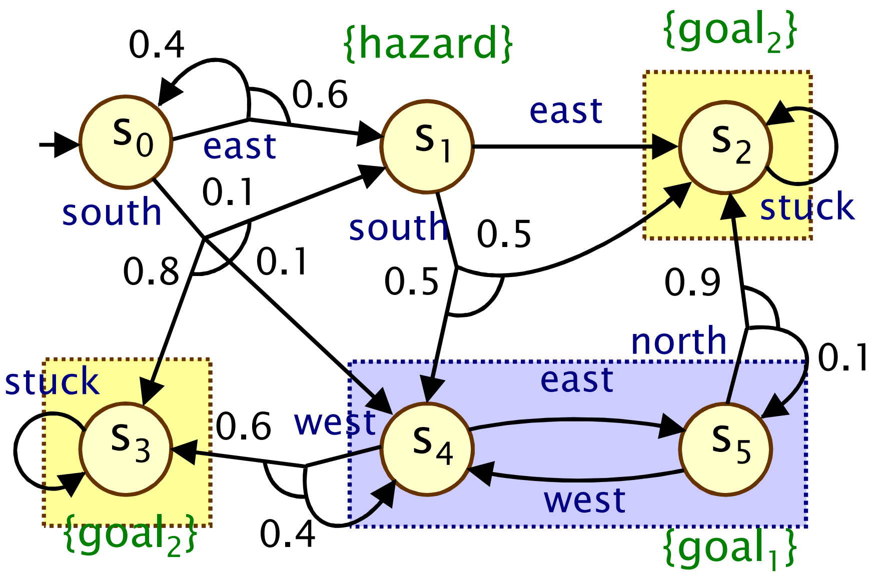
n=1: $[\ \max(0.6 \cdot 0 + 0.4 \cdot 0,\ 0.1 \cdot 0 + 0.1 \cdot 1 + 0.8 \cdot 0),\ \max(0, 0.5),\ 0,\ 0,\ 1,\ 1\ ]$

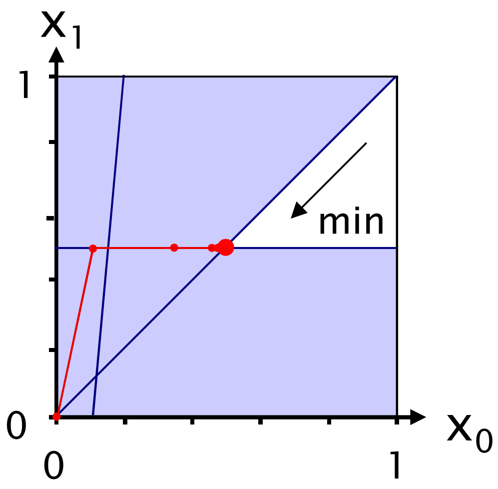
$= [\ 0.1, 0.5, 0, 0, 1, 1\ ]$

n=2: $[\ \max(0.6 \cdot 0.5 + 0.4 \cdot 0.1,\ 0.1 \cdot 0.5 + 0.1 \cdot 1 + 0.8 \cdot 0),\ \max(0, 0.5),\ 0,\ 0,\ 1,\ 1\ ]$

$= [\ 0.34, 0.5, 0, 0, 1, 1\ ]$

Example - Reachability (val. iter.)





$[x_0^{(n)},x_1^{(n)},x_2^{(n)},x_3^{(n)},x_4^{(n)},x_5^{(n)}]$

n=0: [0, 0, 0, 0, 1, 1]

n=1: [0.1, 0.5, 0, 0, 1, 1]

n=2: [0.34, 0.5, 0, 0, 1, 1]

n=3: [0.436, 0.5, 0, 0, 1, 1]

n=4: [0.4744, 0.5, 0, 0, 1, 1]

n=5: [0.48976, 0.5, 0, 0, 1, 1]

n=6: [0.495904, 0.5, 0, 0, 1, 1]

n=7: [0.4983616, 0.5, 0, 0, 1, 1]

n=8: [0.49934464, 0.5, 0, 0, 1, 1]

…

n=16: [0.49999957, 0.5, 0, 0, 1, 1]

n=17: [0.49999982, 0.5, 0, 0, 1, 1]

… $\approx$ [0.5 0.5, 0, 0, 1, 1]

Memoryless strategies

- Memoryless strategies suffice for probabilistic reachability
  - i.e. there exist memoryless strategies σ_{min} & σ_{max} such that:
  - $\mathrm{Prob}^{\sigma_{min}}(s, F\ a) = p_{min}(s, F\ a)$ for all states $s \in S$
  - $\mathrm{Prob}^{\sigma_{max}}(s, F\ a) = p_{max}(s, F\ a)$ for all states $s \in S$
- Construct strategies from optimal solution:

$$\sigma_{min}(s) = \mathrm{argmin} \left\{ \sum_{s' \in S} \mu(s') \cdot p_{min}(s', F\,a) \mid (a, \mu) \in \mathrm{Steps}(s) \right\}$$

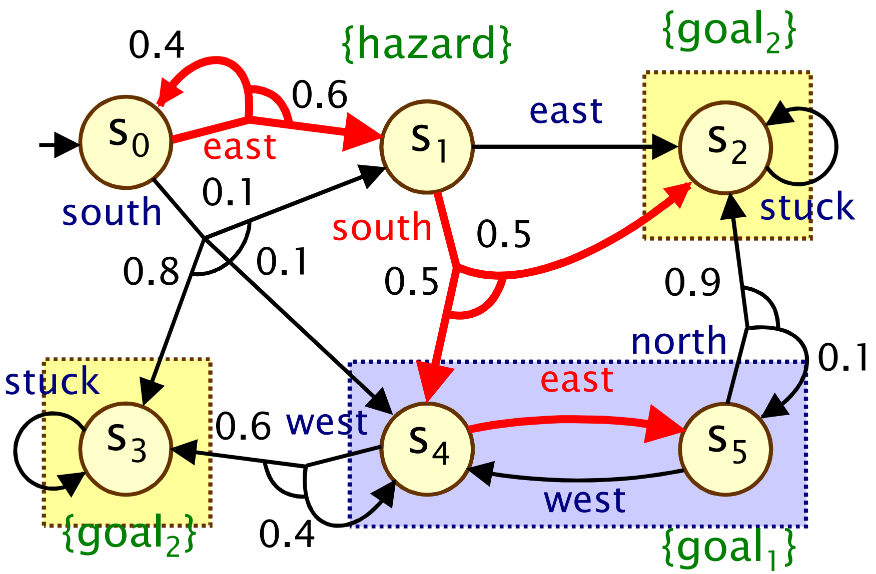
$$\sigma_{max}(s) = \mathrm{argmax} \left\{ \sum_{s' \in S} \mu(s') \cdot p_{max}(s', F\,a) \mid (a, \mu) \in \mathrm{Steps}(s) \right\}$$

Strategy synthesis

- Compute optimal probabilities $Pr_s^{max}(F\ b)$ for all $s \in S$

- To compute the optimal strategy σ^*, choose the locally optimal action in each state
  - in general depends on the method used to compute the optimal probabilities
- For reachability
  - memoryless strategies suffice
- For step-bounded reachability
  - need finite-memory strategies
  - typically requires backward computation for a fixed number of steps

Example – Strategy



Optimal strategy:

s_0 : east

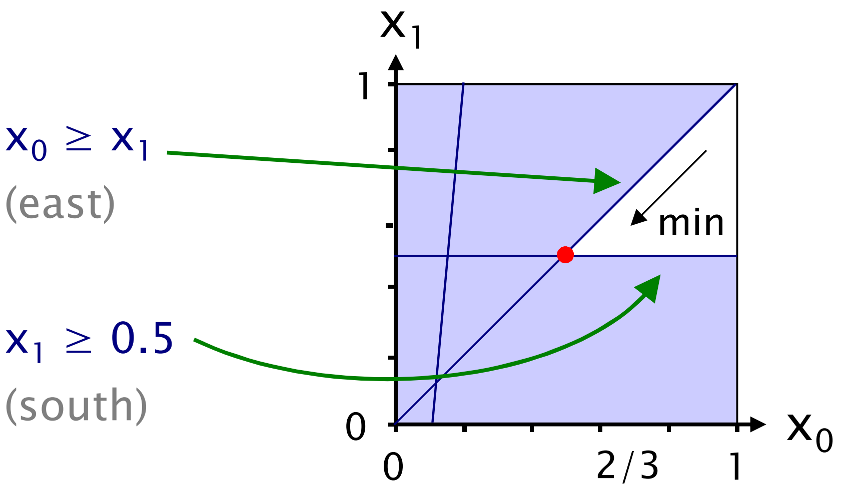
s_1 : south

s_2 : –

s_3 : –

s_4 : east

s_5 : –



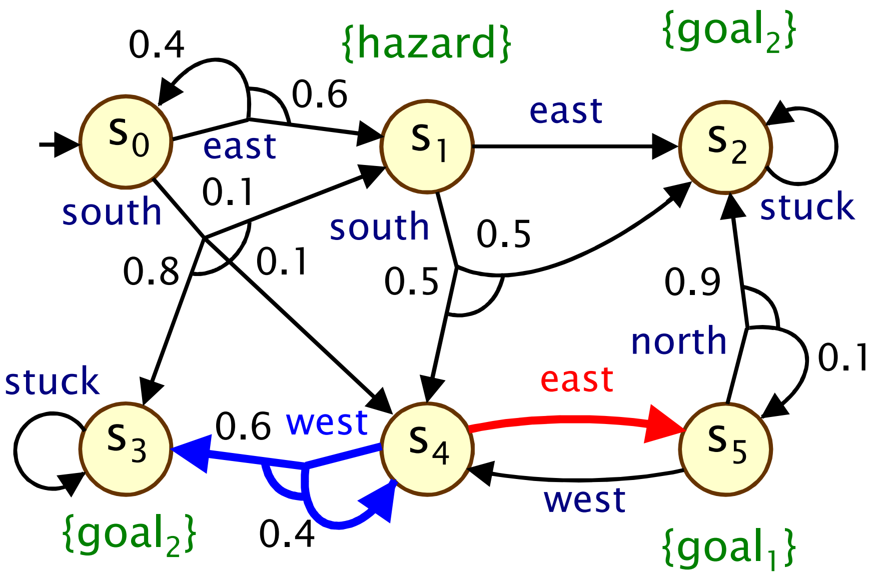
$x_0 \geq x_1$

(east)

$x_1 \geq 0.5$

(south)

Example - Bounded reachability



Example:

$P_{\max=?}$ [$F^{\leq 3}$ goal$_2$]

So compute:

$Pr_s^{\max}(F^{\leq 3}$ goal$_2$) = 0.99

Optimal strategy

is finite-memory:

s_4 (after 1 step): east

s_4 (after 2 steps): west

Strategy synthesis for LTL objectives

- Reduce to the problem of reachability on the product of MDP M and an omega-automaton representing ψ
  - for example, deterministic Rabin automaton (DRA)

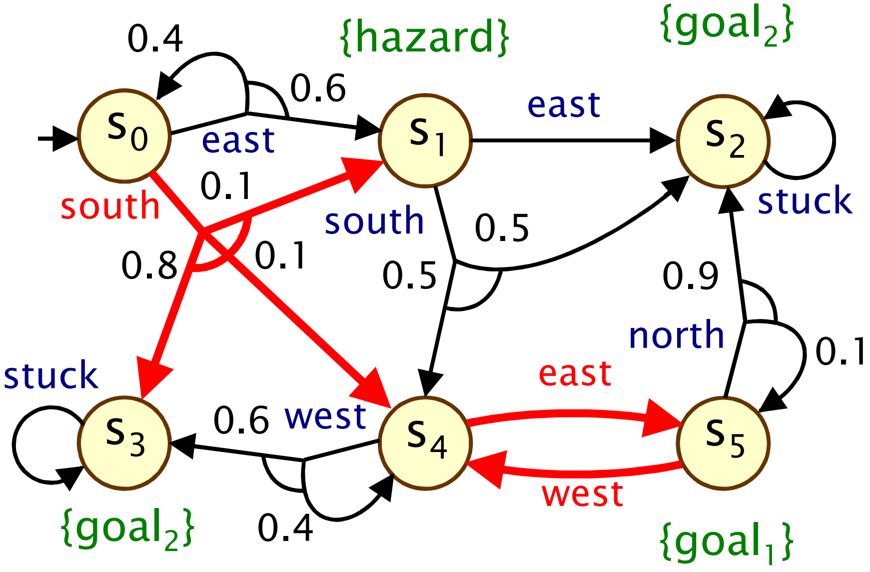
- Need only consider computation of maximum probabilities $Pr_s^{max}(\psi)$
  - since $Pr_s^{min}(\psi) = 1 - Pr_s^{max}(\neg\psi)$

- To compute the optimal strategy σ*
  - find memoryless deterministic strategy on the product
  - convert to finite-memory strategy with one mode for each state of the DRA for ψ

Example – LTL

- $P_{\geq 0.05}$ [(G ¬hazard) ∧ (GF $goal_1$)]
  - avoid hazard and visit $goal_1$ infinitely often

- Pr_{s0}^{max}((G ¬hazard) ∧ (GF $goal_1$)) = 0.1

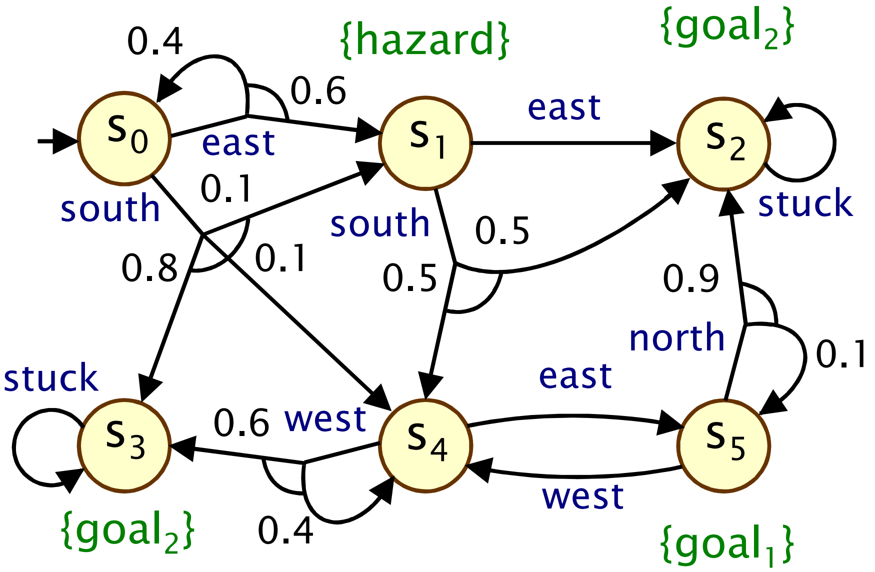


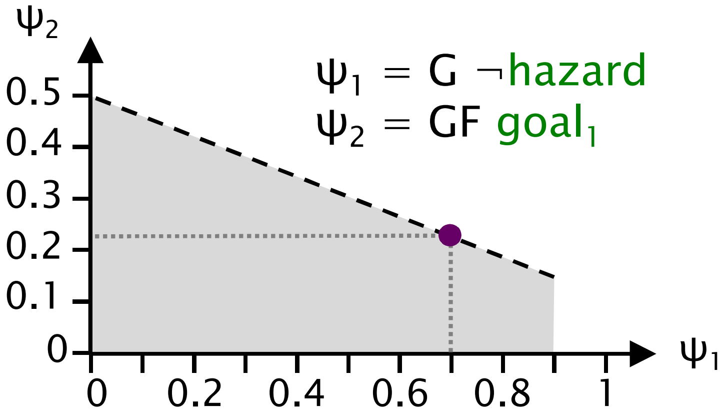
Optimal strategy:
(in this instance,
memoryless)
s_0 : south
s_1 : –
s_2 : –
s_3 : –
s_4 : east
s_5 : west

Multi-objective strategy synthesis

- Consider conjunctions of probabilistic LTL formulas $P_{\sim p}$ [ψ]
  - require all conjuncts to be satisfied
- Reduce to a multi-objective reachability problem on the product of MDP M and the omega-automata representing the conjuncts
  - convert (by negation) to formulas with upper probability bounds (≥, >), then to DRA
  - need to consider all combinations of objectives
- The problem can be solved using LP methods [TACAS07] or via approximations to Pareto curve [ATVA12]
  - strategies may be finite memory and randomised
- Continue as for single-objectives to compute the strategy σ*
  - find memoryless deterministic strategy on the product
  - convert to finite-memory strategy

Example – Multi-objective



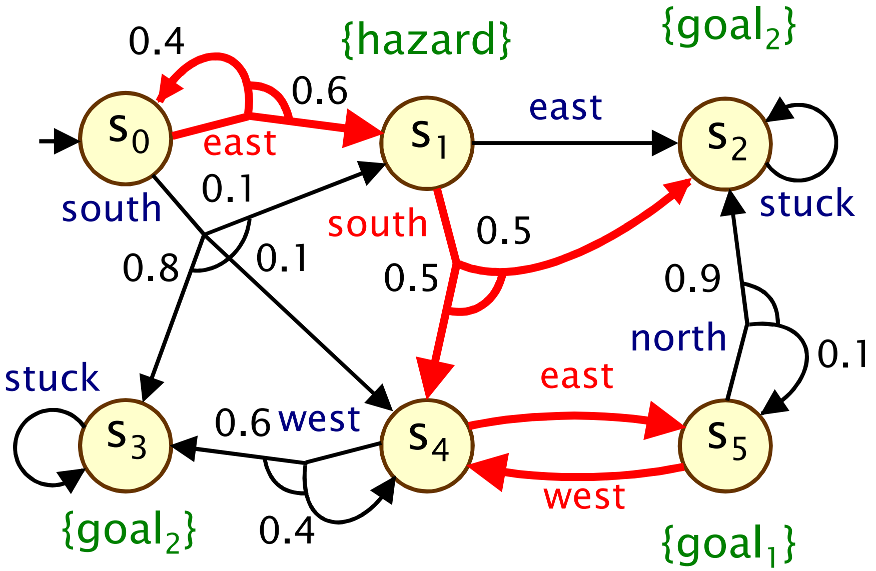


ψ_1 = G ¬hazard

ψ_2 = GF goal$_1$

- Multi-objective formula
  - $P_{\geq 0.7}$ [G ¬hazard] ∧ $P_{\geq 0.2}$ [GF goal$_1$] ? True (achievable)
- Numerical query
  - $P_{\max=?}$ [GF goal$_1$] such that $P_{\geq 0.7}$ [G ¬hazard] ? ~0.2278
- Pareto query
  - for $P_{\max=?}$ [G ¬hazard] ∧ $P_{\max=?}$ [GF goal$_1$] ?

Example – Multi-objective strategies



Strategy 1

(deterministic)

s_0 : east

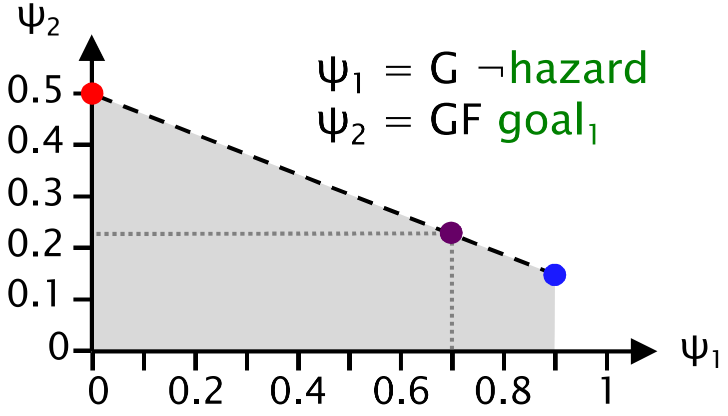
s_1 : south

s_2 : –

s_3 : –

s_4 : east

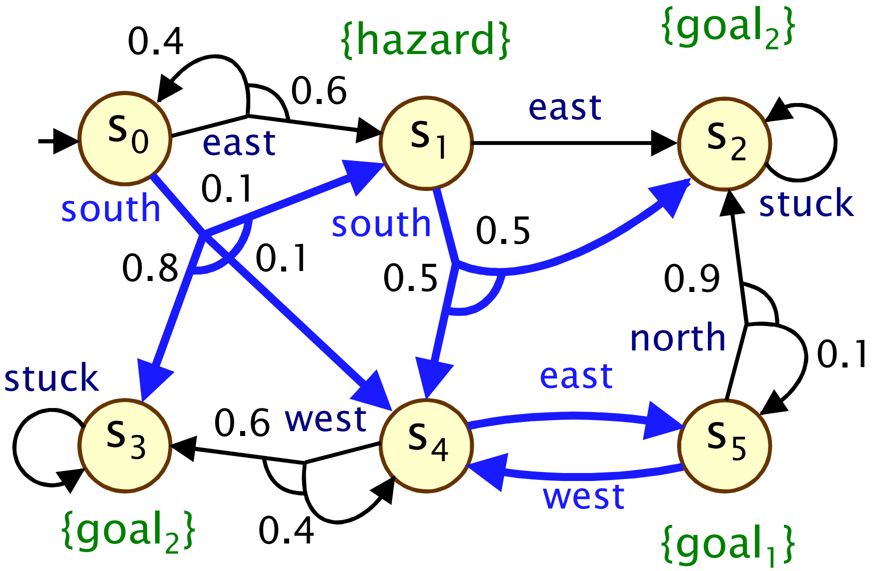
s_5 : west

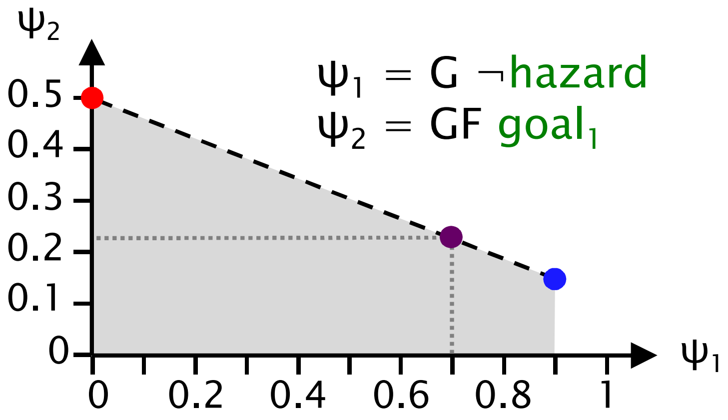


ψ_1 = G ¬hazard

ψ_2 = GF goal$_1$

Example - Multi-objective strategies





ψ_1 = G ¬hazard

ψ_2 = GF goal$_1$

Strategy 2

(deterministic)

s_0 : south

s_1 : south

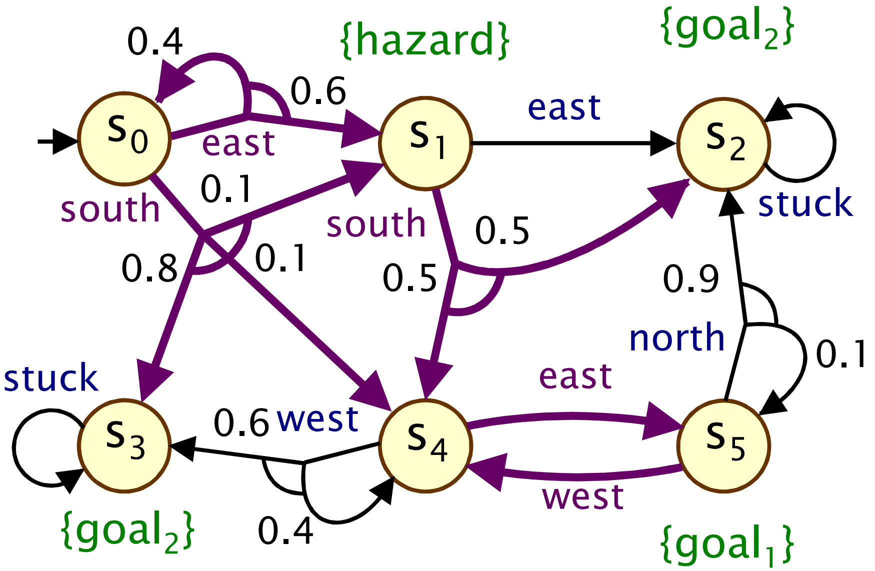
s_2 : –

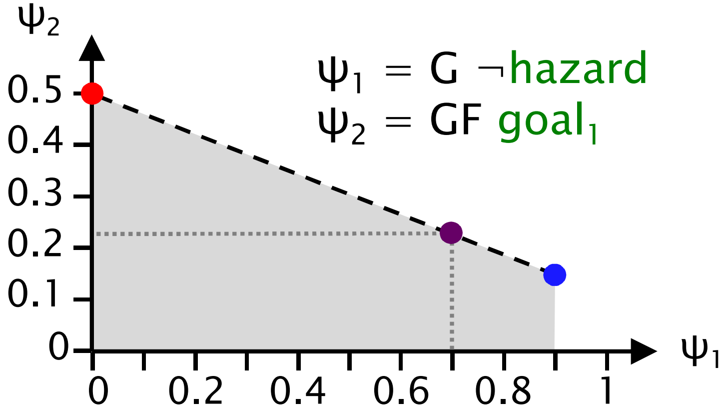
s_3 : –

s_4 : east

s_5 : west

Example – Multi-objective strategies





$\psi_1 = \mathrm{G}\ \neg$hazard

$\psi_2 = \mathrm{GF}$ goal$_1$

Optimal strategy:

(randomised)

s$_0$: 0.3226 : east

0.6774 : south

s$_1$: 1.0 : south

s$_2$: –

s$_3$: –

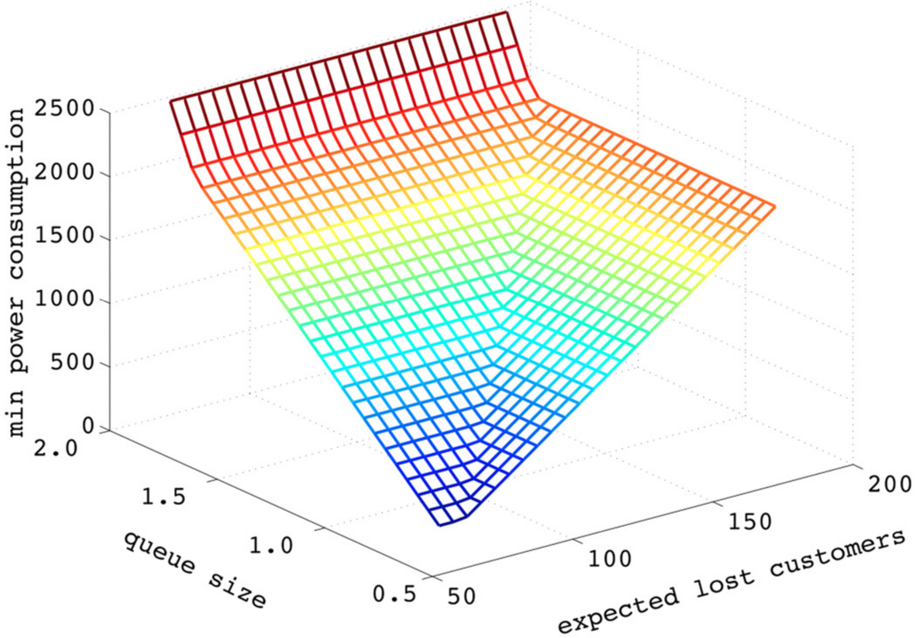
s$_4$: 1.0 : east

s$_5$: 1.0 : west

Case study: Dynamic power management

- Synthesis of dynamic power management schemes
  - for an IBM TravelStar VP disk drive
  - 5 different power modes: active, idle, idlelp, stby, sleep
  - power manager controller bases decisions on current power mode, disk request queue, etc.

- Build controllers that
  - minimise energy consumption, subject to constraints on e.g.
  - probability that a request waits more than K steps
  - expected number of lost disk requests



- See: lab and http://www.prismmodelchecker.org/files/tacas11/

PRISM: Recent & new developments

- New features:
  1. parametric model checking
  2. parameter synthesis
  3. strategy synthesis
  4. stochastic multi-player games
  5. real-time: probabilistic timed automata (PTAs)

- Further new additions:
  - enhanced statistical model checking (approximations + confidence intervals, acceptance sampling)
  - efficient CTMC model checking (fast adaptive uniformisation)
  - benchmark suite & testing functionality
  - www.prismmodelchecker.org

– Beyond PRISM…

Summary (Part 2)

- Markov decision processes (MDPs)
  - extend DTMCs with nondeterminism
  - to model concurrency, underspecification, …
- Property specifications
  - PCTL: exactly same syntax as for DTMCs
  - but quantify over all strategies
- Model checking algorithms
  - covered three basic techniques for MDPs: linear programming, value iteration, or policy iteration
  - introduced multi-objective specifications
- Strategy synthesis
  - can reuse model checking algorithms