Probabilistic model checking with PRISM

Marta Kwiatkowska

Department of Computer Science, University of Oxford

IMT, Lucca, May 2016

What is probabilistic model checking?

- Probabilistic model checking (aka probabilistic/ quantitative verification)…
  - is a formal verification technique for modelling and analysing systems that exhibit probabilistic behaviour
- Formal verification…
  - is the application of rigorous, mathematics-based techniques to establish the correctness of computerised systems
- Synthesis…
  - is an automatic method to generate system components that are correct-by-construction

Why must we verify?

“Testing can only show the presence of errors, not their absence.”

To rule out errors need to consider all possible executions often not feasible mechanically!

- need formal verification…

“In their capacity as a tool, computers will be but a ripple on the surface of our culture. In their capacity as intellectual challenge, computers are without precedent in the cultural history of mankind.”

Edsger Dijkstra

1930-2002

But my program works!

- True, there are many successful large-scale complex computer systems…
  - online banking, electronic commerce
  - information services, online libraries, business processes
  - supply chain management
  - mobile phone networks
- Yet many new potential application domains with far greater complexity and higher expectations
  - autonomous driving, self-parking cars
  - medical sensors: heart rate & blood pressure monitors
  - intelligent buildings and spaces, environmental sensors
- Learning from mistakes costly…

Infusion pumps

F.D.A. Steps Up Oversight of Infusion Pumps

The New York Times Published: April 23, 2010

Pump producers now typically conduct 'simulated' testing of its devices by users.

Over the last five years, […] 710 patient deaths linked to problems with the devices.

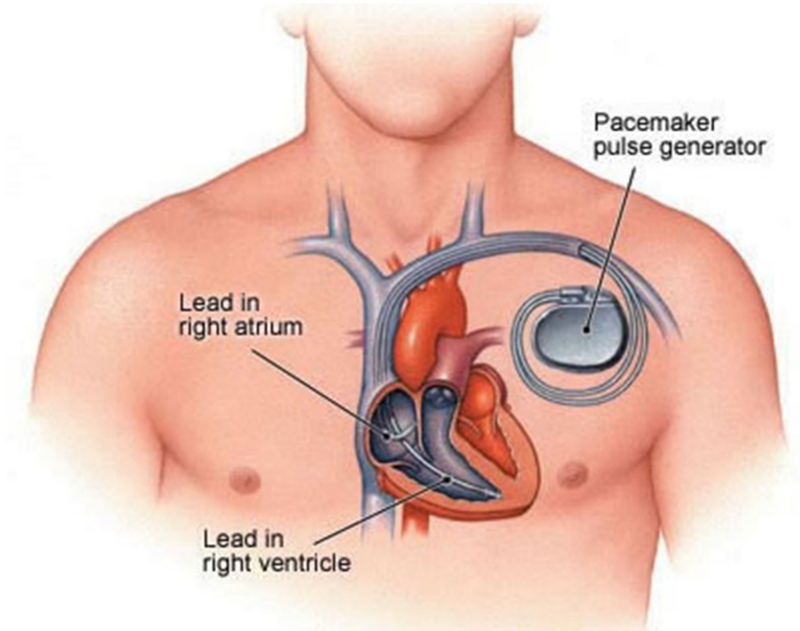
Some of those deaths involved patients who suffered drug overdoses accidentally, either because of incorrect dosage entered or because the device's software malfunctioned.

Manufacturers […] issued 79 recalls, among the highest for any medical device.

Source: http://www.nytimes.com/2010/04/24/business/24pump.html?_r=0

Cardiac pacemakers

- The Food and Drug Administration (FDA)
  - issued 23 recalls of defective pacemaker devices during the first half of 2010
  - classified as “Class I,” meaning there is “reasonable probability that use of these products will cause serious adverse health consequences or death”
  - six of those due to software defects



- “Killed by code” report
  - many similar medical devices
  - wireless, implantable, e.g. glucose monitors

Source: https://www.softwarefreedom.org/resources/2010/transparent-medical-devices.html

Toyota

- February 2010
  - unintended acceleration
  - resulted in deaths

- Engine Control Module
  - source code found defective
  - no mirroring: stack overflow , recursion was used
- “Killed by firmware”
  - millions of cars recalled, at huge cost
  - handling of the incident prompted much criticism, bad publicity
  - fined $1.2 billion for concealing safety defects

Source: http://www.edn.com/design/automotive/4423428/Toyota-s-killer-firmware--Bad-design-and-its-consequences

What do these stories have in common?

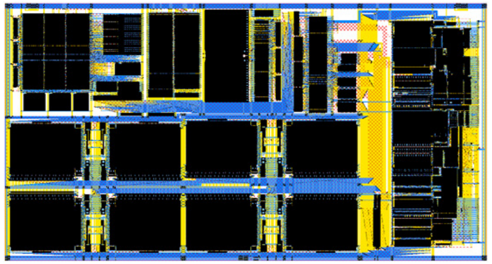
- Programmable computing devices
  - conventional computers and networks
  - software embedded in devices
    - airbag controllers, mobile phones, medical devices, etc
- Programming error direct cause of failure

- Software critical
  - for safety
  - for business
  - for performance
- High costs incurred: not just financial

- Failures avoidable…

Automatic verification

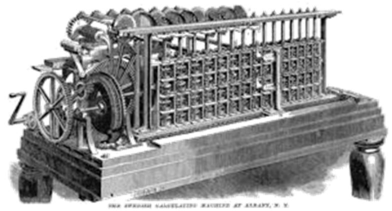
- Formal verification…
  - the application of rigorous, mathematics-based techniques to establish the correctness of computerised systems
  - essentially: proving that a program satisfies it specification
  - many techniques: manual proof, automated theorem proving, static analysis, model checking, …



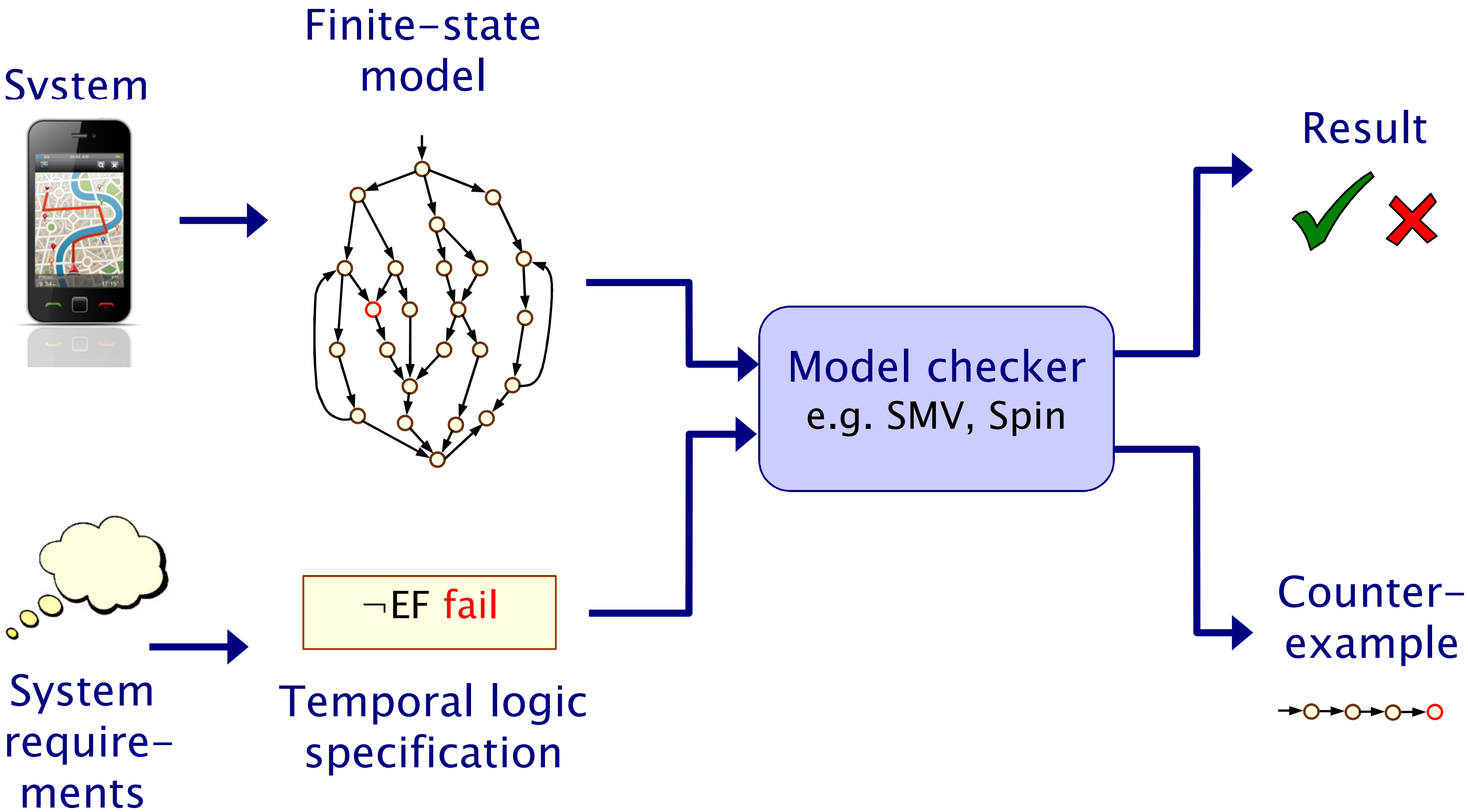
$10^{500,000}$ states

10^{70} atoms

- Automatic verification =
  - mechanical, push-button technology
  - performed without human intervention



Verification via model checking



Verification… or falsification?

- More value in showing property violation?
  - model checkers used as debugging tool!
  - can we synthesise directly from specification?

- Widely accepted in industrial practice
  - Intel, Cadence, Bell Labs, IBM, Microsoft, ...

- Many software tools, including commercial
  - CProver/CBMC, NuSMV, FDR2, UPPAAL, ...
  - hardware design, protocols, software, …

Much progress since 1981! But...

New challenges for verification

- Devices, ever smaller
  - laptops, phones, sensors…

- Networking, wireless, wired & global
  - wireless & internet everywhere

- New design and engineering challenges
  - adaptive computing, ubiquitous/pervasive computing, context-aware systems
  - DNA computing and biosensing
  - trade-offs between e.g. performance, security, power usage, battery life, …

New challenges for verification

- Many properties other than correctness are important
- Need to guarantee…
  - safety, reliability, performance, dependability
  - resource usage, e.g. battery life
  - security, privacy, trust, anonymity, fairness
  - and much more…

- Quantitative, as well as qualitative requirements:
  - "how reliable is my car's Bluetooth network?"
  - "how efficient is my phone's power management policy?"
  - "how secure is my bank's web-service?"

- This course: probabilistic verification and synthesis

Why probability?

- Some systems are inherently probabilistic…

- Randomisation, e.g. in distributed coordination algorithms
  - as a symmetry breaker, in gossip routing to reduce flooding

- Examples: real-world protocols featuring randomisation:
  - Randomised back-off schemes
    - CSMA protocol, 802.11 Wireless LAN
  - Random choice of waiting time
    - IEEE1394 Firewire (root contention), Bluetooth (device discovery)
  - Random choice over a set of possible addresses
    - IPv4 Zeroconf dynamic configuration (link-local addressing)
  - Randomised algorithms for anonymity, contract signing, …

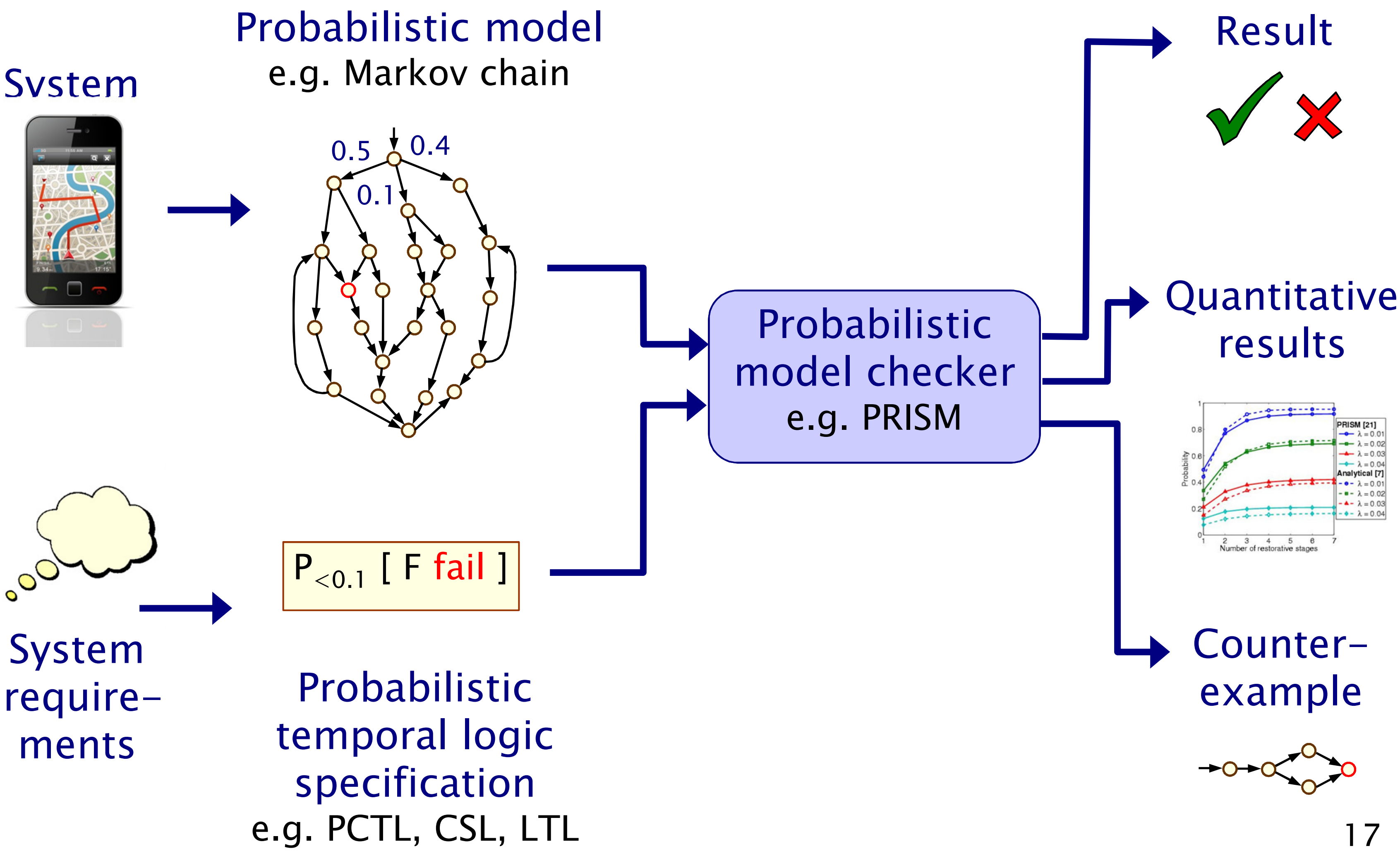
Why probability?

- Some systems are inherently probabilistic…
- Randomisation, e.g. in distributed coordination algorithms
  - as a symmetry breaker, in gossip routing to reduce flooding
- To model uncertainty and performance
  - to quantify rate of failures, express Quality of Service
- Examples:
  - computer networks, embedded systems
  - power management policies
  - nano-scale circuitry: reliability through defect-tolerance

Why probability?

- Some systems are inherently probabilistic…
- Randomisation, e.g. in distributed coordination algorithms
  - as a symmetry breaker, in gossip routing to reduce flooding
- To model uncertainty and performance
  - to quantify rate of failures, express Quality of Service
- To model biological processes
  - reactions occurring between large numbers of molecules are naturally modelled in a stochastic fashion

Probabilistic model checking



Probabilistic models

	Fully probabilistic	Nondeterministic
Discrete time	Discrete-time Markov chains (DTMCs)	Markov decision processes (MDPs)
		Simple stochastic games (SMGs)
Continuous time	Continuous-time Markov chains (CTMCs)	Probabilistic timed automata (PTAs)
		Interactive Markov chains (IMCs)

Probabilistic models

	Fully probabilistic	Nondeterministic
Discrete time	Discrete-time Markov chains (DTMCs)	Markov decision processes (MDPs)
		Simple stochastic games (SMGs)
Continuous time	Continuous-time Markov chains (CTMCs)	Probabilistic timed automata (PTAs)
		Interactive Markov chains (IMCs)

NB One can also consider continuous space…

Lecture plan

- Course slides and lab session
  - http://www.prismmodelchecker.org/courses/imt16/
- 3 sessions: lectures 9-11
  - 1 – Discrete time Markov chains (DTMCs)
  - 2 – Markov decision processes (MDPs)
  - 3 – LTL model checking for DTMCs/MDPs
- For extended versions of this material
  - and an accompanying list of references
  - see: http://www.prismmodelchecker.org/lectures/

Course material

- Reading
  - [DTMCs/CTMCs] Kwiatkowska, Norman and Parker. Stochastic Model Checking. LNCS vol 4486, p220-270, Springer 2007.
  - [MDPs/LTL] Forejt, Kwiatkowska, Norman and Parker. Automated Verification Techniques for Probabilistic Systems. LNCS vol 6659, p53-113, Springer 2011.
  - [SMGs] Chen, Forejt, Kwiatkowska, Parker and Simaitis. Automatic Verification of Competitive Stochastic Systems. FMSD 43(1), 61-92, 2013.
  - [PTAs] Norman, Parker and Sproston. Model Checking for Probabilistic Timed Automata. FMSD 43(2), 164-190, 2013.
  - [DTMCs/MDPs/LTL] Principles of Model Checking by Baier and Katoen, MIT Press 2008
- See also PRISM website
  - www.prismmodelchecker.org

Part 1

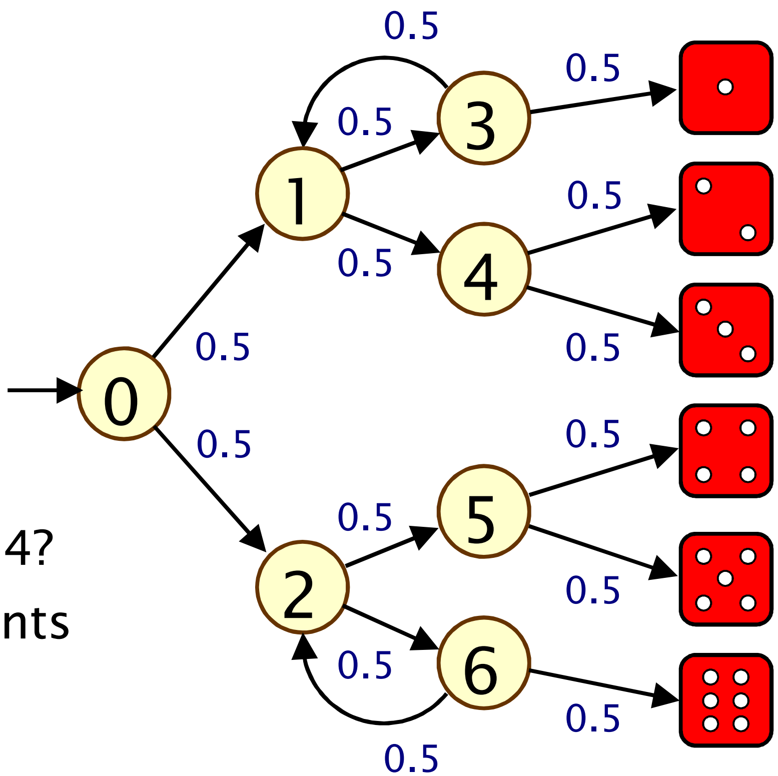
Discrete-time Markov chains

Overview (Part 1)

- Probability basics
- Model checking for discrete-time Markov chains (DTMCs)
  - DTMCs: definition, paths & probability spaces
  - PCTL model checking
  - Costs and rewards
- A glimpse of model checking for continuous-time Markov chains (CTMCs)
- PRISM: overview
  - Modelling language
  - Properties
  - GUI, etc
  - Case study: Bluetooth device discovery
- Summary

Probability example

- Modelling a 6-sided die using a fair coin
  - algorithm due to Knuth/Yao:
  - start at 0, toss a coin
  - upper branch when H
  - lower branch when T
  - repeat until value chosen

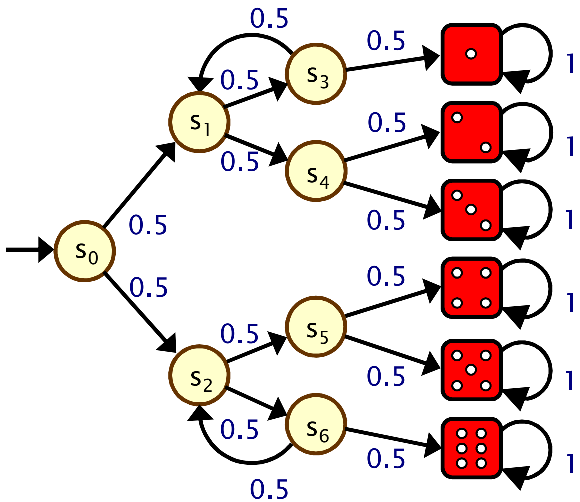


- Is this algorithm correct?
  - e.g. probability of obtaining a 4?
  - obtain as disjoint union of events
  - THH, TTTHH, TTTTTHH, …
  - Pr("eventually 4")

    $= (1/2)^3 + (1/2)^5 + (1/2)^7 + \ldots = 1/6$

Example...

- Other properties?
  - "what is the probability of termination?"
- e.g. efficiency?
  - "what is the probability of needing more than 4 coin tosses?"
  - "on average, how many coin tosses are needed?"

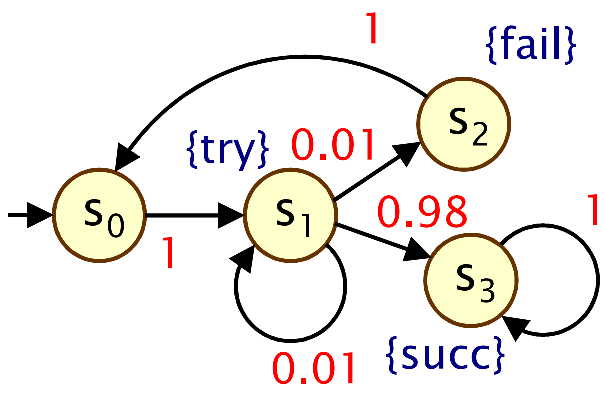


- Probabilistic model checking provides a framework for these kinds of properties...
  - modelling languages
  - property specification languages
  - model checking algorithms, techniques and tools

Discrete-time Markov chains

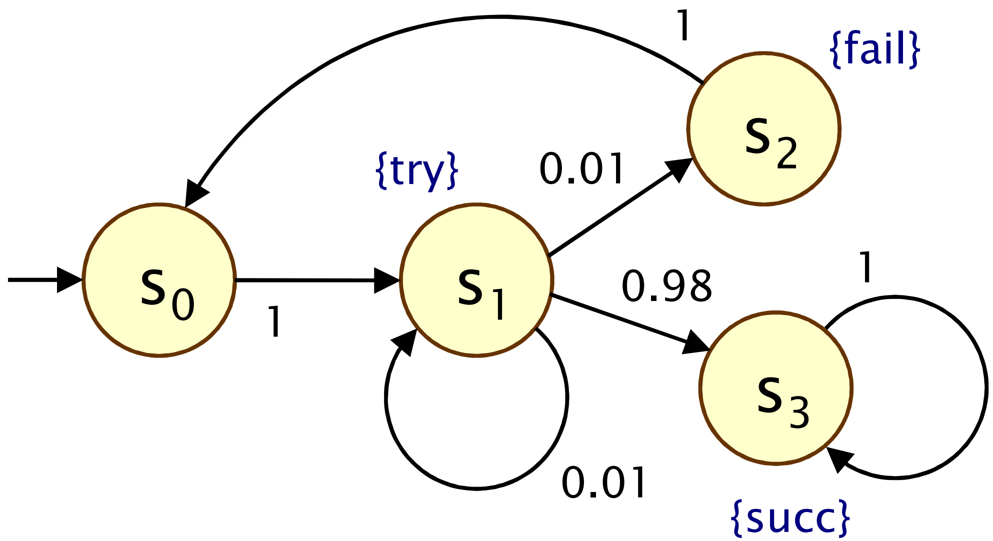
- Discrete-time Markov chains (DTMCs)
  - state-transition systems augmented with probabilities

- States
  - discrete set of states representing possible configurations of the system being modelled
- Transitions
  - transitions between states occur in discrete time-steps
- Probabilities
  - probability of making transitions between states is given by discrete probability distributions



Simple DTMC example

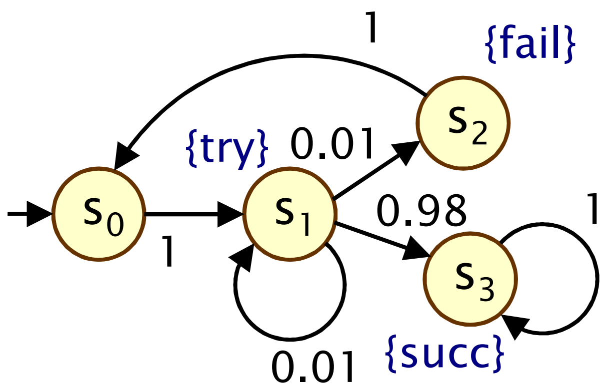
- Modelling a very simple communication protocol
  - after one step, process starts trying to send a message
  - with probability 0.01, channel unready so wait a step
  - with probability 0.98, send message successfully and stop
  - with probability 0.01, message sending fails, restart



Discrete-time Markov chains

- Formally, a DTMC D is a tuple (S, s_{init}, P, L) where:
  - S is a finite set of states ("state space")
  - $s_{init} \in S$ is the initial state
  - $P : S \times S \rightarrow [0,1]$ is the transition probability matrix where $\Sigma_{s' \in S} P(s,s') = 1$ for all $s \in S$
  - $L : S \rightarrow 2^{AP}$ is function labelling states with atomic propositions

- Note: no deadlock states
  - i.e. every state has at least one outgoing transition
  - can add self loops to represent final/terminating states



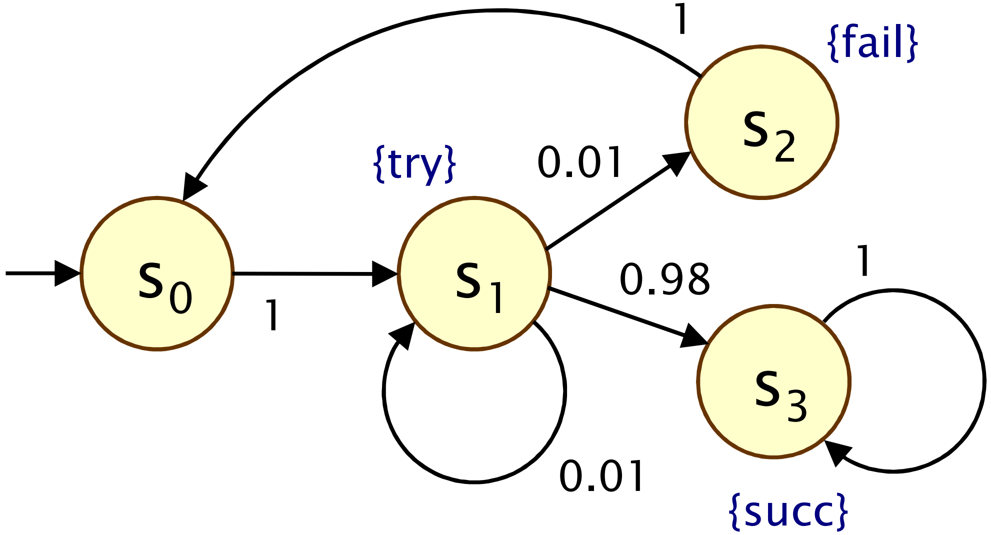
Simple DTMC example

$D = (S, s_{init}, P, L)$

$S = \{s_0, s_1, s_2, s_3\}$
$s_{init} = s_0$

$AP = \{try, fail, succ\}$
$L(s_0) = \emptyset$,
$L(s_1) = \{try\}$,
$L(s_2) = \{fail\}$,
$L(s_3) = \{succ\}$

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0.01 & 0.01 & 0.98 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Some more terminology

- P is a stochastic matrix, meaning it satisifes:
  - $P(s,s') \in [0,1]$ for all $s,s' \in S$ and $\Sigma_{s' \in S} P(s,s') = 1$ for all $s \in S$

- A sub-stochastic matrix satisfies:
  - $P(s,s') \in [0,1]$ for all $s,s' \in S$ and $\Sigma_{s' \in S} P(s,s') \leq 1$ for all $s \in S$

- An absorbing state is a state s for which:
  - $P(s,s) = 1$ and $P(s,s') = 0$ for all $s \neq s'$
  - the transition from s to itself is sometimes called a self-loop

- Note: Since we assume P is stochastic…
  - every state has at least one outgoing transition
  - i.e. no deadlocks (in model checking terminology)

DTMCs: An alternative definition

- Alternative definition… a DTMC is:
  - a family of random variables { X(k) | k=0,1,2,… }
  - where X(k) are observations at discrete time-steps
  - i.e. X(k) is the state of the system at time-step k
  - which satisfies…

- The Markov property ("memorylessness")
  - $Pr(X(k)=s_k \mid X(k-1)=s_{k-1}, \dots , X(0)=s_0)$
    $= Pr(X(k)=s_k \mid X(k-1)=s_{k-1})$
  - for a given current state, future states are independent of past

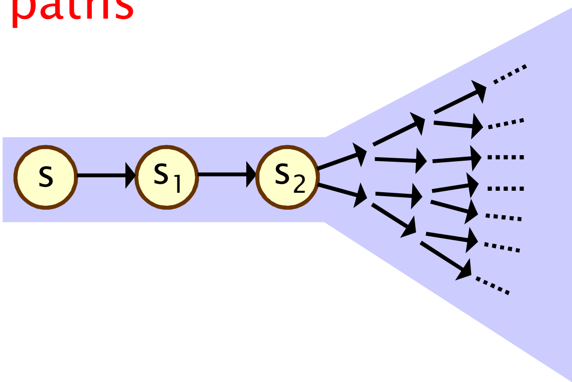
- This allows us to adopt the "state-based" view presented so far (which is better suited to this context)

Other assumptions made here

- We consider time-homogenous DTMCs
  - transition probabilities are independent of time
  - $P(s_{k-1},s_k) = Pr(X(k)=s_k \mid X(k-1)=s_{k-1})$
  - otherwise: time-inhomogenous
- We will (mostly) assume that the state space S is finite
  - in general, S can be any countable set
- Initial state $s_{init} \in S$ can be generalised…
  - to an initial probability distribution $s_{init} : S \rightarrow [0,1]$
- Focus on path-based properties
  - rather than steady-state

Paths and probabilities

- A (finite or infinite) path through a DTMC
  - is a sequence of states $s_0s_1s_2s_3\ldots$ such that $P(s_i,s_{i+1}) > 0\ \forall i$
  - represents an execution (i.e. one possible behaviour) of the system which the DTMC is modelling
- To reason (quantitatively) about this system
  - need to define a probability space over paths
- Intuitively:
  - sample space: Path(s) = set of all infinite paths from a state s



  - events: sets of infinite paths from s
  - basic events: cylinder sets (or "cones")
  - cylinder set C(ω), for a finite path ω = set of infinite paths with the common finite prefix ω
  - for example: $C(ss_1s_2)$

Probability spaces

- Let Ω be an arbitrary non-empty set
- A σ-algebra (or σ-field) on Ω is a family Σ of subsets of Ω closed under complementation and countable union, i.e.:
  - if $A \in \Sigma$, the complement $\Omega \setminus A$ is in Σ
  - if $A_i \in \Sigma$ for $i \in \mathbb{N}$, the union $\cup_i A_i$ is in Σ
  - the empty set $\varnothing$ is in Σ
- Theorem: For any family F of subsets of Ω, there exists a unique smallest σ-algebra on Ω containing F
- Probability space (Ω, Σ, Pr)
  - Ω is the sample space
  - Σ is the set of events: σ-algebra on Ω
  - $Pr : \Sigma \to [0,1]$ is the probability measure:

    $Pr(\Omega) = 1$ and $Pr(\cup_i A_i) = \Sigma_i Pr(A_i)$ for countable disjoint A_i

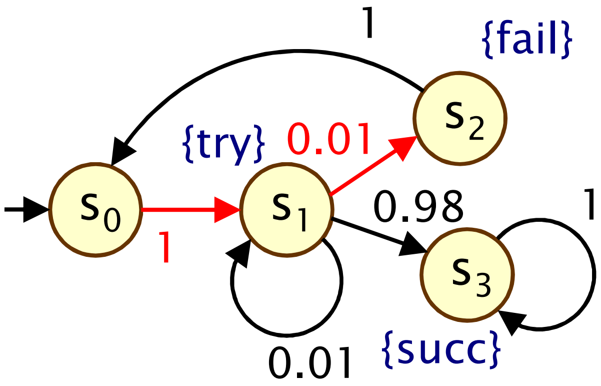
Probability space over paths

- Sample space Ω = Path(s)
  set of infinite paths with initial state s
- Event set $\Sigma_{Path(s)}$
  - the cylinder set $C(\omega) = \{ \omega' \in Path(s) \mid \omega \text{ is prefix of } \omega' \}$
  - $\Sigma_{Path(s)}$ is the least σ-algebra on Path(s) containing C(ω) for all finite paths ω starting in s
- Probability measure Pr_s
  - define probability $P_s(\omega)$ for finite path $\omega = ss_1 \dots s_n$ as:
    - $P_s(\omega) = 1$ if ω has length one (i.e. ω = s)
    - $P_s(\omega) = P(s,s_1) \cdot \dots \cdot P(s_{n-1},s_n)$ otherwise
    - define $Pr_s(C(\omega)) = P_s(\omega)$ for all finite paths ω
  - Pr_s extends uniquely to a probability measure $Pr_s : \Sigma_{Path(s)} \to [0,1]$

- See [KSK76] for further details

Probability space – Example

- Paths where sending fails the first time
  - $\omega = s_0 s_1 s_2$
  - $C(\omega)$ = all paths starting $s_0 s_1 s_2 \ldots$
  - $P_{s0}(\omega) = P(s_0,s_1) \cdot P(s_1,s_2)$
    $= 1 \cdot 0.01 = 0.01$
  - $Pr_{s0}(C(\omega)) = P_{s0}(\omega) = 0.01$



- Paths which are eventually successful and with no failures
  - $C(s_0 s_1 s_3) \cup C(s_0 s_1 s_1 s_3) \cup C(s_0 s_1 s_1 s_1 s_3) \cup \ldots$
  - $Pr_{s0}(\, C(s_0 s_1 s_3) \cup C(s_0 s_1 s_1 s_3) \cup C(s_0 s_1 s_1 s_1 s_3) \cup \ldots \,)$
    $= P_{s0}(s_0 s_1 s_3) + P_{s0}(s_0 s_1 s_1 s_3) + P_{s0}(s_0 s_1 s_1 s_1 s_3) + \ldots$
    $= 1 \cdot 0.98 + 1 \cdot 0.01 \cdot 0.98 + 1 \cdot 0.01 \cdot 0.01 \cdot 0.98 + \ldots$
    $= 0.9898989898\ldots$
    $= 98/99$

Reachability

- Key property: probabilistic reachability
  - probability of a path reaching a state in some target set $T \subseteq S$
  - e.g. “probability of the algorithm terminating successfully?”
  - e.g. “probability that an error occurs during execution?”

- Dual of reachability: invariance
  - probability of remaining within some class of states
  - Pr(“remain in set of states T”) = 1 - Pr(“reach set $S \backslash T$”)
  - e.g. “probability that an error never occurs”

- We will also consider other variants of reachability
  - time-bounded, constrained (“until”), …

Reachability probabilities

- Formally: ProbReach(s, T) = Pr_s(Reach(s, T))
  - where Reach(s, T) = { $s_0 s_1 s_2 \ldots \in$ Path(s) | s_i in T for some i }

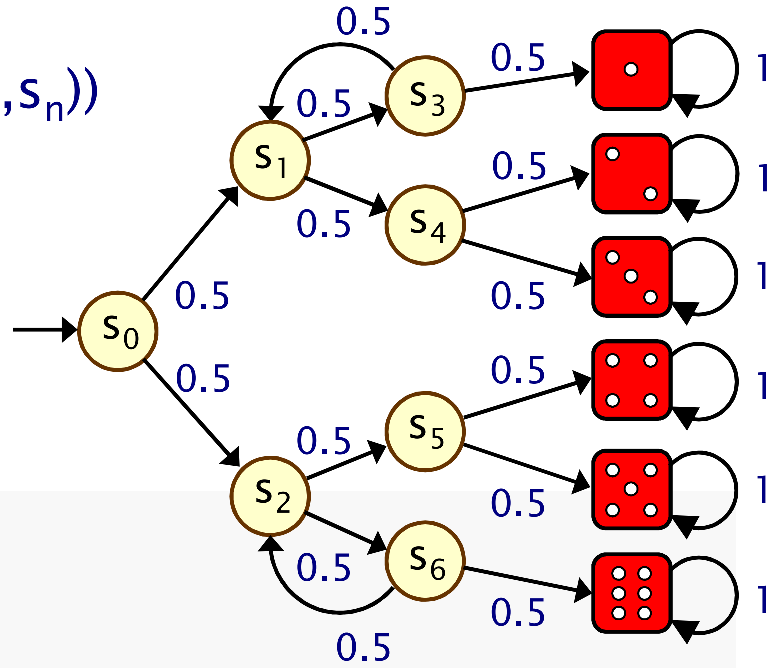
- Is Reach(s, T) measurable for any $T \subseteq S$? Yes…
  - Reach(s, T) is the union of all basic cylinders Cyl($s_0 s_1 \ldots s_n$) where $s_0 s_1 \ldots s_n$ in Reach_{fin}(s, T)
  - Reach_{fin}(s, T) contains all finite paths $s_0 s_1 \ldots s_n$ such that: $s_0 = s$, $s_0, \ldots, s_{n-1} \notin T$, $s_n \in T$ (reaches T first time)
  - set of such finite paths $s_0 s_1 \ldots s_n$ is countable

- Probability
  - in fact, the above is a disjoint union
  - so probability obtained by simply summing…

Computing reachability probabilities

- Compute as (infinite) sum…
- $\Sigma_{s_0,\ldots,s_n \in \text{Reachfin}(s,\ T)} \Pr_{s_0}(\text{Cyl}(s_0,\ldots,s_n))$

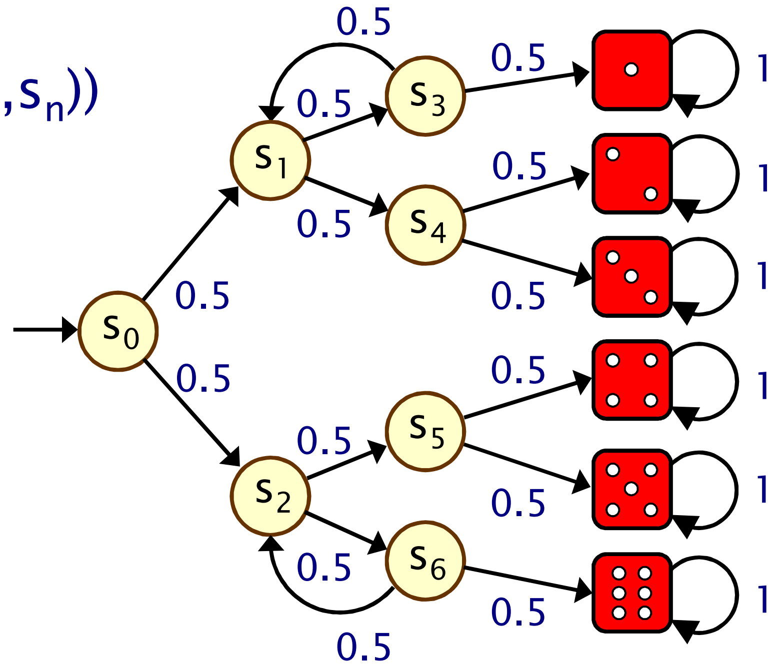
  $= \Sigma_{s_0,\ldots,s_n \in \text{Reachfin}(s,\ T)} \mathbf{P}(s_0,\ldots,s_n)$
- Example:
  - ProbReach(s_0, {4})



Computing reachability probabilities

- Compute as (infinite) sum…
- $\Sigma_{s0,\ldots,sn \,\in\, \text{Reachfin(s, T)}}\ \mathrm{Pr}_{s0}(\mathrm{Cyl}(s_0,\ldots,s_n))$

  $= \Sigma_{s0,\ldots,sn \,\in\, \text{Reachfin(s, T)}}\ \mathbf{P}(s_0,\ldots,s_n)$



- Example:
  - ProbReach(s_0, {4})
  - = Pr_{s0}(Reach(s_0, {4}))
  - Finite path fragments:
  - $s_0(s_2s_6)^n s_2s_5 4$ for $n \geq 0$
  - $\mathbf{P}_{s0}(s_0s_2s_54) + \mathbf{P}_{s0}(s_0s_2s_6s_2s_54) + \mathbf{P}_{s0}(s_0s_2s_6s_2s_6s_2s_54) + \ldots$

  $= (1/2)^3 + (1/2)^5 + (1/2)^7 + \ldots = 1/6$

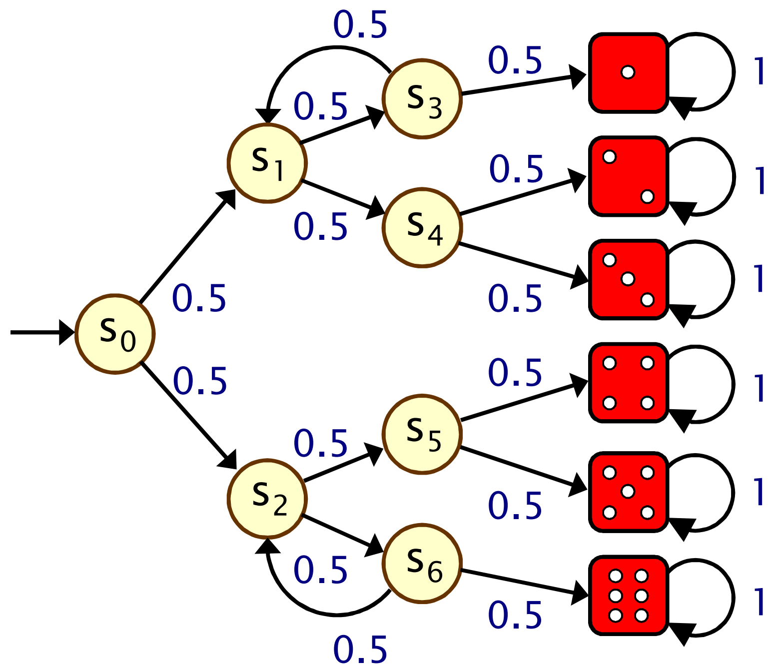
Computing reachability probabilities

- Alternative: derive a linear equation system
  - solve for all states simultaneously
  - i.e. compute vector $\underline{\text{ProbReach}}(T)$
- Let x_s denote ProbReach(s, T)
- Solve:

$$x_s = \begin{cases} 1 & \text{if } s \in T \\ 0 & \text{if } T \text{ is not reachable from } s \\ \sum_{s' \in S} P(s,s') \cdot x_{s'} & \text{otherwise} \end{cases}$$

Example

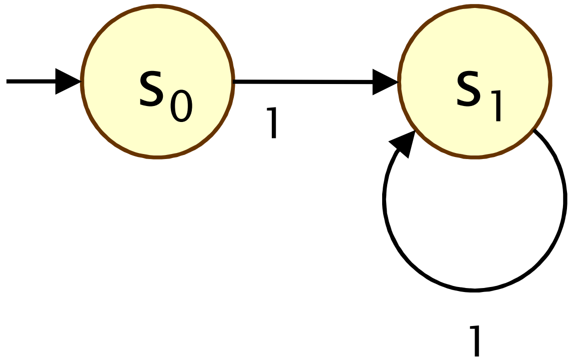
- Compute ProbReach(s_0, {4})



Unique solutions

- Why the need to identify states that cannot reach T?

- Consider this simple DTMC:
  - compute probability of reaching $\{s_0\}$ from s_1



  - linear equation system: $x_{s_0} = 1$, $x_{s_1} = x_{s_1}$
  - multiple solutions: $(x_{s_0}, x_{s_1}) = (1,p)$ for any $p \in [0,1]$

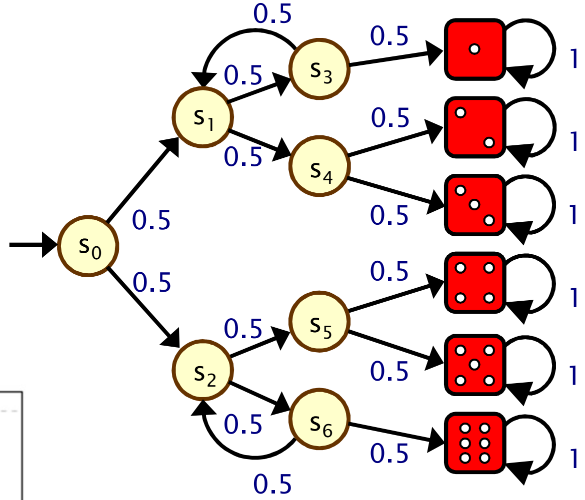
Bounded reachability probabilities

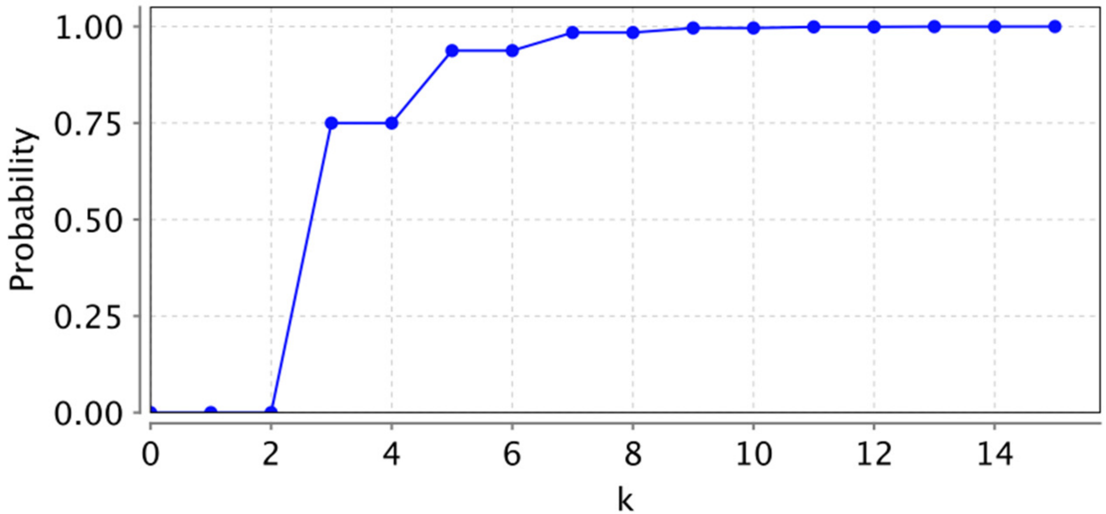
- Probability of reaching T from s within k steps
- Formally: $\mathrm{ProbReach}^{\leq k}(s, T) = \Pr_s(\mathrm{Reach}^{\leq k}(s, T))$ where:
  - $\mathrm{Reach}^{\leq k}(s, T) = \{ s_0 s_1 s_2 \ldots \in \mathrm{Path}(s) \mid s_i \text{ in } T \text{ for some } i \leq k \}$
- $\underline{\mathrm{ProbReach}}^{\leq k}(T) = \underline{x}^{(k+1)}$ from the previous fixed point
  - which gives us…

$$\mathrm{ProbReach}^{\leq k}(s, T) = \begin{cases} 1 & \text{if } s \in T \\ 0 & \text{if } k = 0 \;\&\; s \notin T \\ \sum_{s' \in S} P(s,s') \cdot \mathrm{ProbReach}^{\leq k-1}(s', T) & \text{if } k > 0 \;\&\; s \notin T \end{cases}$$

(Bounded) reachability

- ProbReach(s_0, {1,2,3,4,5,6}) = 1
- ProbReach$^{\leq k}$ (s_0, {1,2,3,4,5,6}) = …





Qualitative properties

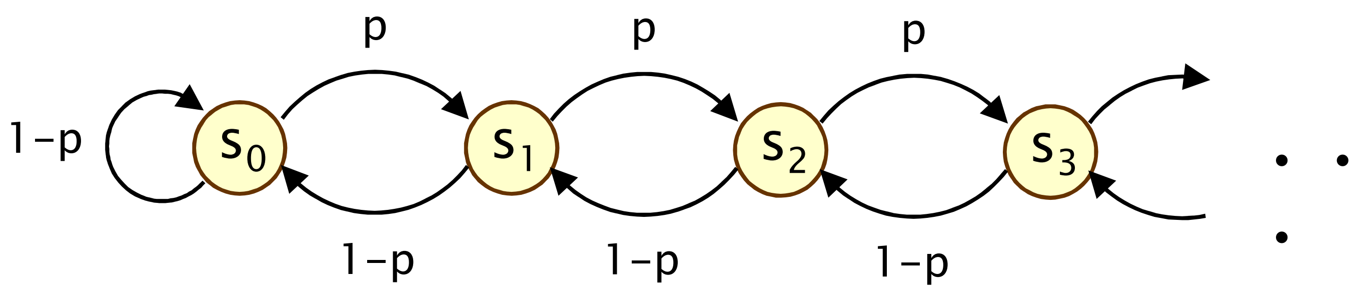
- Quantitative properties:
  - "what is the probability of event A?"

- Qualititative properties:
  - "the probability of event A is 1" ("almost surely A")
  - or: "the probability of event A is > 0" ("possibly A")

- For finite DTMCs, qualititative properties do not depend on the transition probabilities - only need underlying graph
  - e.g. to determine "is target set T reached with probability 1?" (see DTMC model checking later)

Aside: Infinite Markov chains

- Infinite-state random walk

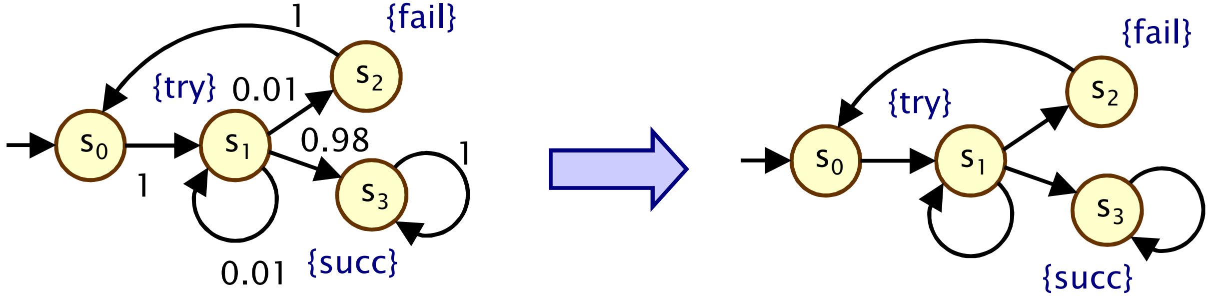


- Value of probability p does affect qualitative properties
  - ProbReach(s, $\{s_0\}$) = 1 if $p \leq 0.5$
  - ProbReach(s, $\{s_0\}$) < 1 if $p > 0.5$

Temporal logic

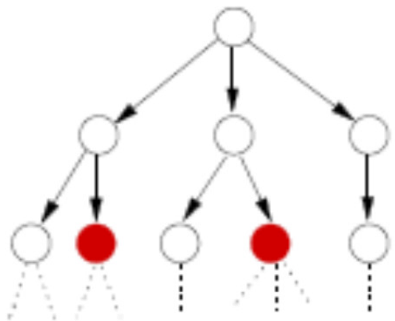
- Temporal logic
  - formal language for specifying and reasoning about how the behaviour of a system changes over time
  - defined over paths, i.e. sequences of states $s_0s_1s_2s_3\ldots$ such that $P(s_i,s_{i+1}) > 0\ \forall i$

- Logics used in this course are probabilistic extensions of temporal logics devised for non-probabilistic systems (CTL, LTL)
  - So we revert briefly to (labelled) state-transition diagrams

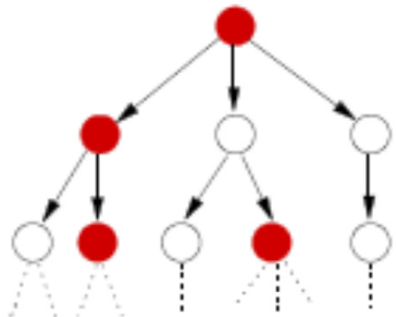


CTL semantics

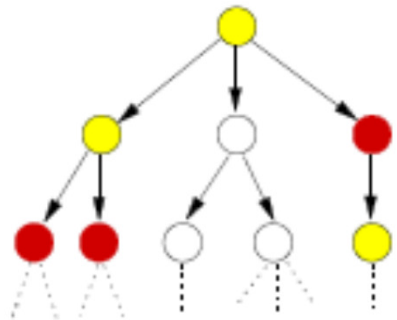
- Intuitive semantics:
  - of quantifiers (A/E) and temporal operators (F/G/U)



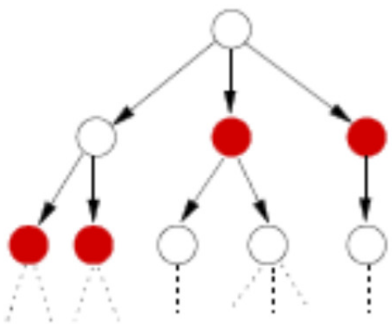
EF red



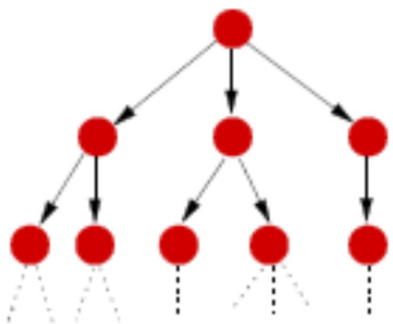
EG red



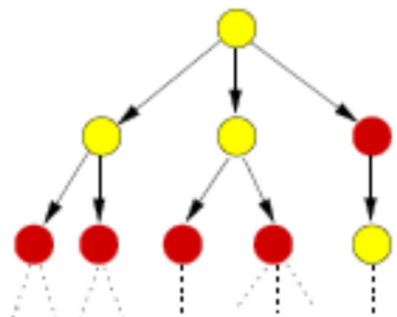
E [yellow U red]



AF red



AG red



A [yellow U red]

PCTL

- Temporal logic for describing properties of DTMCs
  - PCTL = Probabilistic Computation Tree Logic [HJ94]
  - essentially the same as the logic pCTL of [ASB+95]
- Extension of (non-probabilistic) temporal logic CTL
  - key addition is probabilistic operator P
  - quantitative extension of CTL's A and E operators
- Example
  - send $\rightarrow$ $P_{\geq 0.95}$ [true $U^{\leq 10}$ deliver]
  - "if a message is sent, then the probability of it being delivered within 10 steps is at least 0.95"

PCTL syntax

- PCTL syntax:

ψ is true with probability ~p

  - $\phi \ ::= \ \text{true} \mid a \mid \phi \wedge \phi \mid \neg\phi \mid P_{\sim p}\,[\,\psi\,]$ (state formulas)
  - $\psi \ ::= \ X\,\phi \ \mid \ \phi\ U^{\leq k}\ \phi \ \mid \ \phi\ U\ \phi$ (path formulas)

"next" "bounded until" "until"

  - define $F\,\phi \equiv \text{true}\ U\ \phi$ (eventually), $G\,\phi \equiv \neg(F\,\neg\phi)$ (globally)
  - where a is an atomic proposition, used to identify states of interest, $p \in [0,1]$ is a probability, $\sim \in \{<,>,\leq,\geq\}$, $k \in \mathbb{N}$

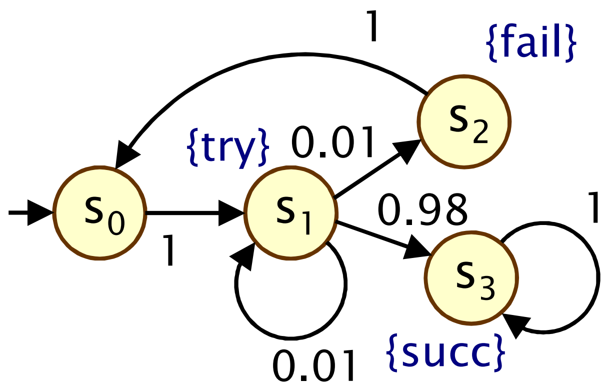
- A PCTL formula is always a state formula
  - path formulas only occur inside the P operator

PCTL semantics for DTMCs

- PCTL formulas interpreted over states of a DTMC
  - $s \vDash \phi$ denotes ϕ is "true in state s" or "satisfied in state s"

- Semantics of (non-probabilistic) state formulas:
  - for a state s of the DTMC (S,s_{init},P,L):
  - $s \vDash a \quad \Leftrightarrow \quad a \in L(s)$
  - $s \vDash \phi_1 \wedge \phi_2 \quad \Leftrightarrow \quad s \vDash \phi_1$ and $s \vDash \phi_2$
  - $s \vDash \neg\phi \quad \Leftrightarrow \quad s \vDash \phi$ is false

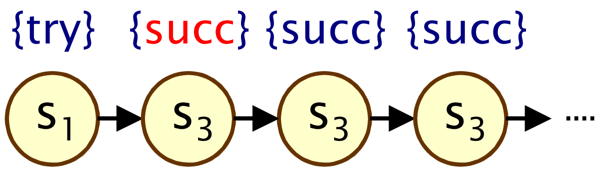
- Examples
  - $s_3 \vDash succ$
  - $s_1 \vDash try \wedge \neg fail$



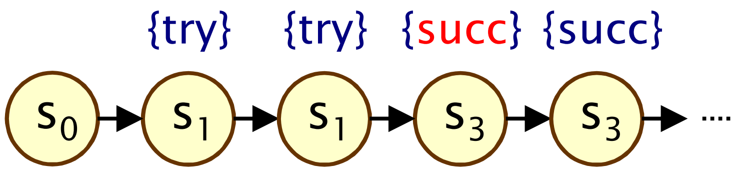
PCTL semantics for DTMCs

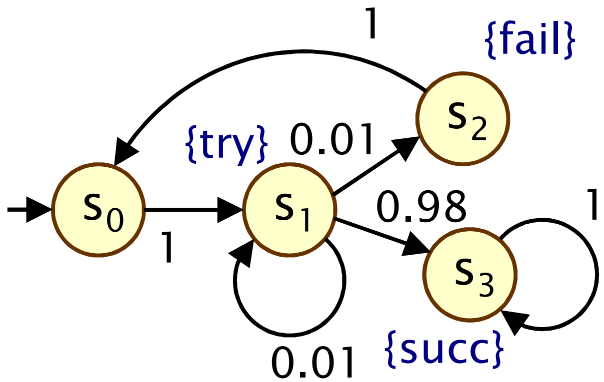
- Semantics of path formulas:
  - for a path $\omega = s_0 s_1 s_2 \ldots$ in the DTMC:
  - $\omega \vDash X\ \phi \quad \Leftrightarrow \quad s_1 \vDash \phi$
  - $\omega \vDash \phi_1\ U^{\leq k}\ \phi_2 \quad \Leftrightarrow \quad \exists i \leq k$ such that $s_i \vDash \phi_2$ and $\forall j<i,\ s_j \vDash \phi_1$
  - $\omega \vDash \phi_1\ U\ \phi_2 \quad \Leftrightarrow \quad \exists k \geq 0$ such that $\omega \vDash \phi_1\ U^{\leq k}\ \phi_2$

- Some examples of satisfying paths:
  - X succ



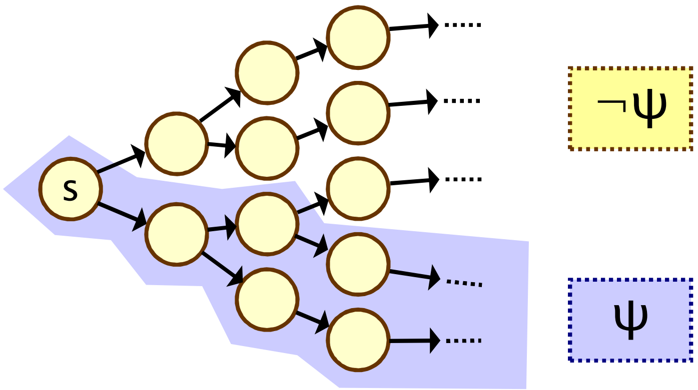
  - ¬fail U succ





PCTL semantics for DTMCs

- Semantics of the probabilistic operator P
  - informal definition: $s \vDash P_{\sim p} [\ \psi\ ]$ means that "the probability, from state s, that ψ is true for an outgoing path satisfies ~p"
  - example: $s \vDash P_{<0.25} [\ X\ fail\ ] \Leftrightarrow$ "the probability of atomic proposition fail being true in the next state of outgoing paths from s is less than 0.25"
  - formally: $s \vDash P_{\sim p} [\psi] \;\Leftrightarrow\; Prob(s, \psi) \sim p$
  - where: $Prob(s, \psi) = Pr_s \{ \omega \in Path(s) \mid \omega \vDash \psi \}$
  - (sets of paths satisfying ψ are always measurable [Var85])



Prob(s, ψ) ~ p ?

More PCTL…

- Usual temporal logic equivalences:
  - false $\equiv \neg$true (false)
  - $\phi_1 \vee \phi_2 \equiv \neg(\neg\phi_1 \wedge \neg\phi_2)$ (disjunction)
  - $\phi_1 \rightarrow \phi_2 \equiv \neg\phi_1 \vee \phi_2$ (implication)

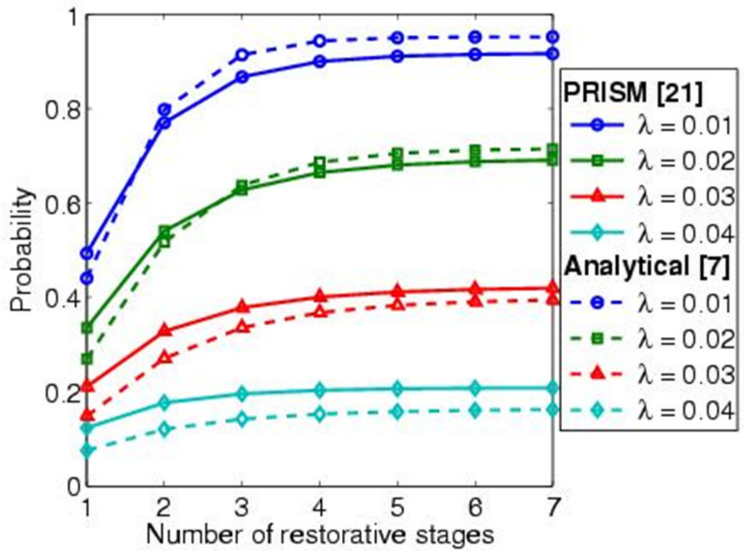
  - $F\ \phi \equiv \Diamond\ \phi \equiv$ true $U\ \phi$ (eventually, “future”)
  - $G\ \phi \equiv \Box\ \phi \equiv \neg(F\ \neg\phi)$ (always, “globally”)
  - bounded variants: $F^{\leq k}\ \phi$, $G^{\leq k}\ \phi$

- Negation and probabilities
  - e.g. $\neg P_{>p}\ [\ \phi_1\ U\ \phi_2\ ] \equiv P_{\leq p}\ [\phi_1\ U\ \phi_2\ ]$
  - e.g. $P_{>p}\ [\ G\ \phi\ ] \equiv P_{<1-p}\ [\ F\ \neg\phi\ ]$

Quantitative properties

- Consider a PCTL formula $P_{\sim p}$ [ψ]
  - if the probability is unknown, how to choose the bound p?
- When the outermost operator of a PTCL formula is P
  - we allow the form $P_{=?}$ [ψ]
  - "what is the probability that path formula ψ is true?"
- Model checking is no harder: compute the values anyway
- Useful to spot patterns, trends

- Example
  - $P_{=?}$ [F err/total>0.1]
  - "what is the probability that 10% of the NAND gate outputs are erroneous?"



Reachability and invariance

- Derived temporal operators, like CTL…

- Probabilistic reachability: $P_{\sim p}$ [F ϕ]
  - the probability of reaching a state satisfying ϕ
  - $F\ \phi \equiv \text{true}\ U\ \phi$
  - "ϕ is eventually true"
  - bounded version: $F^{\leq k}\ \phi \equiv \text{true}\ U^{\leq k}\ \phi$

- Probabilistic invariance: $P_{\sim p}$ [G ϕ]
  - the probability of ϕ always remaining true
  - $G\ \phi \equiv \neg(F\ \neg\phi) \equiv \neg(\text{true}\ U\ \neg\phi)$
  - "ϕ is always true"
  - bounded version: $G^{\leq k}\ \phi \equiv \neg(F^{\leq k}\ \neg\phi)$

strictly speaking, G ϕ cannot be derived from the PCTL syntax in this way since there is no negation of path formulae

Qualitative vs. quantitative properties

- P operator of PCTL can be seen as a quantitative analogue of the CTL operators A (for all) and E (there exists)

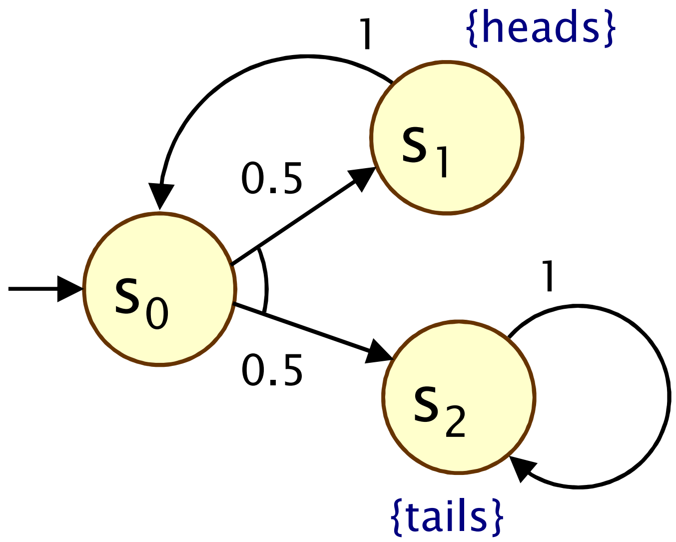
- Qualitative PCTL properties
  - $P_{\sim p}$ [ψ] where p is either 0 or 1
- Quantitative PCTL properties
  - $P_{\sim p}$ [ψ] where p is in the range (0,1)

- $P_{>0}$ [F ϕ] is identical to EF ϕ
  - there exists a finite path to a ϕ-state
- $P_{\geq 1}$ [F ϕ] is (similar to but) weaker than AF ϕ
  - a ϕ-state is reached “almost surely”
  - see next slide…

Example: Qualitative/quantitative

- Toss a coin repeatedly until "tails" is thrown

- Is "tails" always eventually thrown?
  - CTL: AF "tails"
  - Result: false
  - Counterexample: $s_0 s_1 s_0 s_1 s_0 s_1 \ldots$



- Does the probability of eventually throwing "tails" equal one?
  - PCTL: $P_{\geq 1}$ [F "tails"]
  - Result: true
  - Infinite path $s_0 s_1 s_0 s_1 s_0 s_1 \ldots$ has zero probability

Overview (Part 1)

- Probability basics
- Model checking for discrete-time Markov chains (DTMCs)
  - DTMCs: definition, paths & probability spaces
  - PCTL model checking
  - Costs and rewards
- A glimpse of model checking for continuous-time Markov chains (CTMCs)
- PRISM: overview
  - Modelling language
  - Properties
  - GUI, etc
  - Case study: Bluetooth device discovery
- Summary

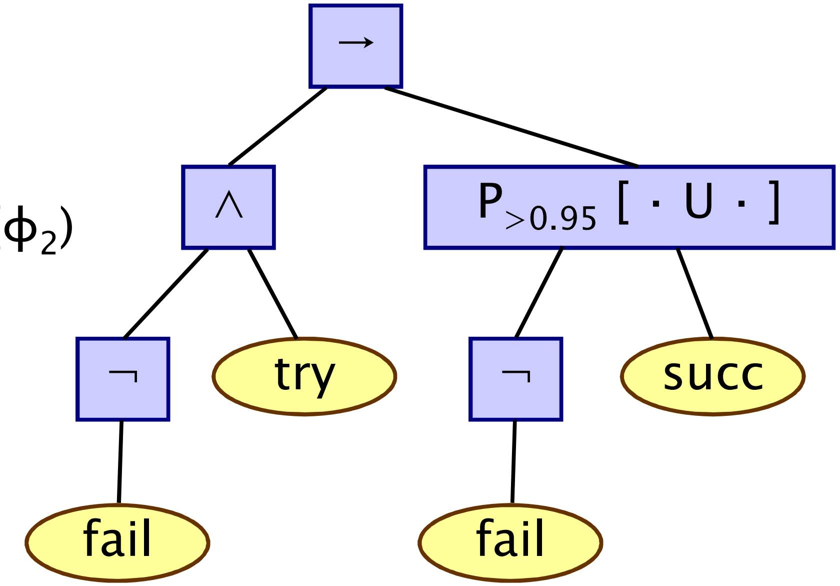
PCTL model checking for DTMCs

- Algorithm for PCTL model checking [CY88,HJ94,CY95]
  - inputs: DTMC $D=(S,s_{init},P,L)$, PCTL formula ϕ
  - output: $Sat(\phi) = \{ s \in S \mid s \vDash \phi \}$ = set of states satisfying ϕ
- What does it mean for a DTMC D to satisfy a formula ϕ?
  - sometimes, want to check that $s \vDash \phi\ \forall\ s \in S$, i.e. $Sat(\phi) = S$
  - sometimes, just want to know if $s_{init} \vDash \phi$, i.e. if $s_{init} \in Sat(\phi)$
- Sometimes, focus on quantitative results
  - e.g. compute result of P=? [F error]
  - e.g. compute result of P=? [$F^{\leq k}$ error] for $0 \leq k \leq 100$

PCTL model checking for DTMCs

- Basic algorithm proceeds by induction on parse tree of ϕ
  - example: $\phi = (\neg\text{fail} \wedge \text{try}) \rightarrow P_{>0.95}$ [$\neg$fail U succ]

- For the non-probabilistic operators:
  - Sat(true) = S
  - Sat(a) = $\{ s \in S \mid a \in L(s) \}$
  - Sat($\neg\phi$) = S \ Sat(ϕ)
  - Sat($\phi_1 \wedge \phi_2$) = Sat(ϕ_1) $\cap$ Sat(ϕ_2)



- For the $P_{\sim p}$ [ψ] operator
  - need to compute the probabilities Prob(s, ψ) for all states $s \in S$
  - focus here on "until" case: $\psi = \phi_1 \text{ U } \phi_2$

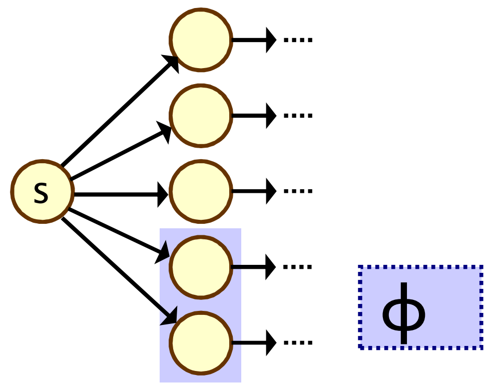
Probability computation

- Three temporal operators to consider:
- Next: $P_{\sim p}[\ X\ \phi\ ]$
- Bounded until: $P_{\sim p}[\ \phi_1\ U^{\leq k}\ \phi_2\ ]$ (omitted)
  - adaptation of bounded reachability for DTMCs
- Until: $P_{\sim p}[\ \phi_1\ U\ \phi_2\ ]$
  - adaptation of reachability for DTMCs
  - graph-based “precomputation” algorithms
  - techniques for solving large linear equation systems

PCTL next for DTMCs

- Computation of probabilities for PCTL next operator
  - $Sat(P_{\sim p}[\ X\ \phi\ ]) = \{\ s \in S \mid Prob(s, X\ \phi) \sim p\ \}$
  - need to compute $Prob(s, X\ \phi)$ for all $s \in S$

- Sum outgoing probabilities for transitions to ϕ-states
  - $Prob(s, X\ \phi) = \Sigma_{s' \in Sat(\phi)}\ P(s,s')$

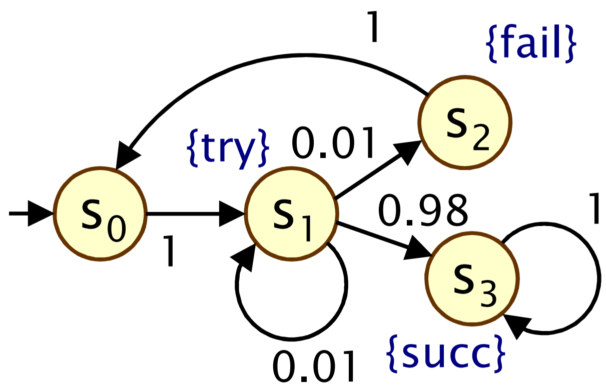


- Compute vector $\underline{Prob}(X\ \phi)$ of probabilities for all states s
  - $\underline{Prob}(X\ \phi) = P \cdot \underline{\phi}$
  - where $\underline{\phi}$ is a 0-1 vector over S with $\underline{\phi}(s) = 1$ iff $s \vDash \phi$
  - computation requires a single matrix-vector multiplication

PCTL next – Example

- Model check: $P_{\geq 0.9}$ [X (¬try ∨ succ)]
  - Sat (¬try ∨ succ) = (S \ Sat(try)) ∪ Sat(succ)
    = ({s_0,s_1,s_2,s_3} \ {s_1}) ∪ {s_3} = {s_0,s_2,s_3}
  - <u>Prob</u>(X (¬try ∨ succ)) = P · <u>(¬try ∨ succ)</u> = …

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0.01 & 0.01 & 0.98 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.99 \\ 1 \\ 1 \end{bmatrix}$$



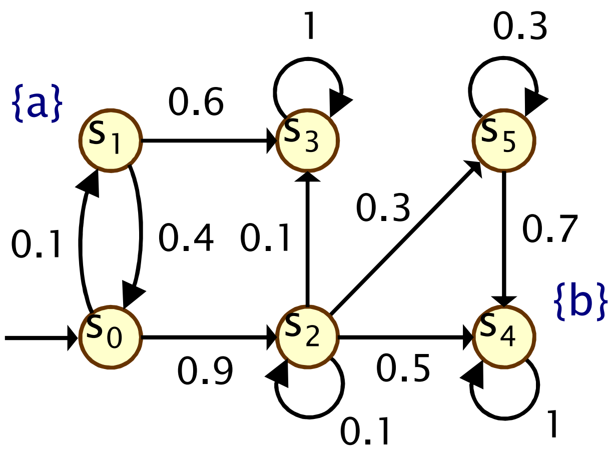
- Results:
  - <u>Prob</u>(X (¬try ∨ succ)) = [0, 0.99, 1, 1]
  - Sat($P_{\geq 0.9}$ [X (¬try ∨ succ)]) = {s_1, s_2, s_3}

PCTL until for DTMCs

- Computation of probabilities Prob(s, ϕ_1 U ϕ_2) for all s $\in$ S
- First, identify all states where the probability is 1 or 0
  - $S^{yes} = Sat(P_{\geq 1}[\ \phi_1\ U\ \phi_2\ ])$
  - $S^{no} = Sat(P_{\leq 0}[\ \phi_1\ U\ \phi_2\ ])$
- Then solve linear equation system for remaining states

- Running example:

$P_{>0.8}\ [\neg a\ U\ b\ ]$



Precomputation

- We refer to the first phase (identifying sets S^{yes} and S^{no}) as "precomputation"
  - two algorithms: Prob0 (for S^{no}) and Prob1 (for S^{yes})
  - algorithms work on underlying graph (probabilities irrelevant)

- Important for several reasons
  - ensures unique solution to linear equation system
    - only need Prob0 for uniqueness, Prob1 is optional
  - reduces the set of states for which probabilities must be computed numerically
  - gives exact results for the states in S^{yes} and S^{no} (no round-off)
  - for model checking of qualitative properties ($P_{\sim p}[\cdot]$ where p is 0 or 1), no further computation required

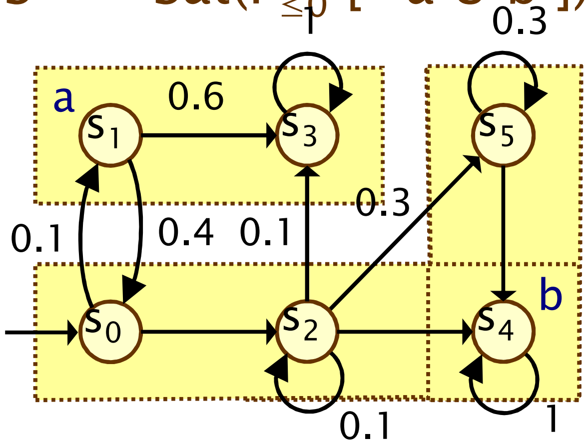
Precomputation – Prob0

- Prob0 algorithm to compute $S^{no} = Sat(P_{\leq 0} [\ \phi_1\ U\ \phi_2\ ])$:
  - first compute $Sat(P_{>0} [\ \phi_1\ U\ \phi_2\ ]) \equiv Sat(E[\ \phi_1\ U\ \phi_2\ ])$
  - i.e. find all states which can, with non-zero probability, reach a ϕ_2-state without leaving ϕ_1-states
  - i.e. find all states from which there is a finite path through ϕ_1-states to a ϕ_2-state: simple graph-based computation
  - subtract the resulting set from S

$S^{no} = Sat(P_{\leq 0} [\neg a\ U\ b\ ])$

Example:

$P_{>0.8} [\neg a\ U\ b\ ]$



Sat(b)

$Sat(P_{>0} [\neg a\ U\ b\ ])$

Precomputation – Prob1

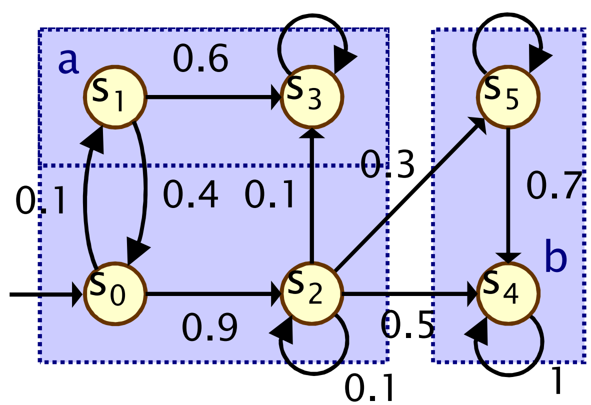
- Prob1 algorithm to compute S^{yes} = Sat($P_{\geq 1}$ [ϕ_1 U ϕ_2]) :
  - first compute Sat($P_{<1}$ [ϕ_1 U ϕ_2]), reusing S^{no}
  - this is equivalent to the set of states which have a non-zero probability of reaching S^{no}, passing only through ϕ_1-states
  - again, this is a simple graph-based computation
  - subtract the resulting set from S

Example:

$P_{>0.8}$ [¬a U b]

S^{no} = Sat($P_{\leq 0}$ [¬a U b])

Sat($P_{<1}$ [¬a U b])



S^{yes} =

Sat($P_{\geq 1}$ [¬a U b])

PCTL until – linear equations

- Probabilities Prob(s, ϕ_1 U ϕ_2) can now be obtained as the unique solution of the following set of linear equations
  - essentially the same as for probabilistic reachability

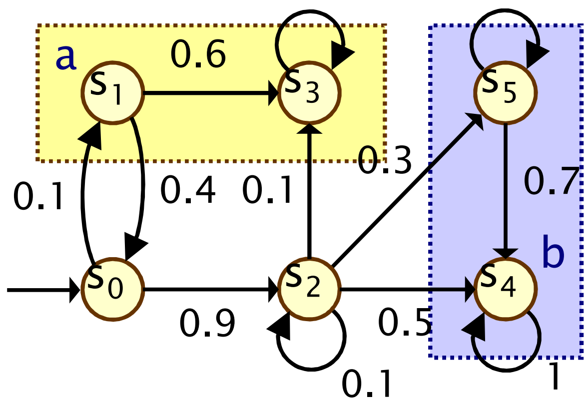
$$\mathrm{Prob}(s, \phi_1 \mathrel{U} \phi_2) = \begin{cases} 1 & \text{if } s \in S^{yes} \\ 0 & \text{if } s \in S^{no} \\ \sum_{s' \in S} P(s,s') \cdot \mathrm{Prob}(s', \phi_1 \mathrel{U} \phi_2) & \text{otherwise} \end{cases}$$

- Can also be reduced to a system in $|S^?|$ unknowns instead of $|S|$ where $S^? = S \setminus (S^{yes} \cup S^{no})$

PCTL until – linear equations

- Example: $P_{>0.8}$ [¬a U b]
- Let x_i = Prob(s_i, ¬a U b)

S^{no} = Sat($P_{\leq 0}$ [¬a U b])



S^{yes} = Sat($P_{\geq 1}$ [¬a U b])

$x_1 = x_3 = 0$

$x_4 = x_5 = 1$

$x_2 = 0.1x_2+0.1x_3+0.3x_5+0.5x_4 = 8/9$

$x_0 = 0.1x_1+0.9x_2 = 0.8$

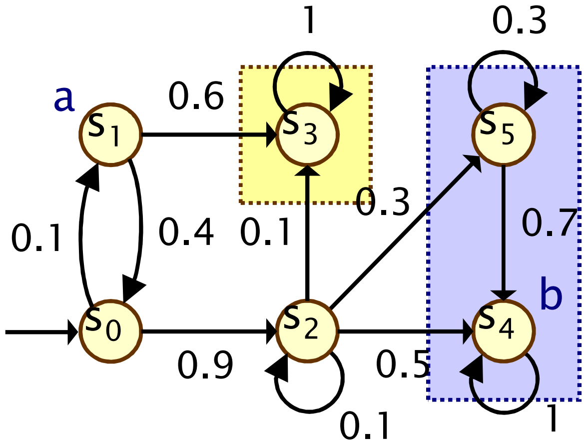
<u>Prob</u>(¬a U b) = <u>x</u> = [0.8, 0, 8/9, 0, 1, 1]

Sat($P_{>0.8}$ [¬a U b]) = { s_2,s_4,s_5 }

PCTL Until – Example 2

- Example: $P_{>0.5}$ [G¬b]
- Prob(s_i, G¬b)
  = 1 – Prob(s_i, ¬(G¬b))
  = 1 – Prob(s_i, F b)
- Let x_i = Prob(s_i, F b)

S^{no} = Sat($P_{\leq 0}$ [F b])



S^{yes} = Sat($P_{\geq 1}$ [F b])

x_3 = 0 and $x_4 = x_5$ = 1

$x_2 = 0.1x_2+0.1x_3+0.3x_5+0.5x_4 = 8/9$

$x_1 = 0.6x_3+0.4x_0 = 0.4x_0$

$x_0 = 0.1x_1+0.9x_2 = 5/6$ and $x_1 = 1/3$

Prob(G¬b) = $\underline{1}-\underline{x}$ = [1/6, 2/3, 1/9, 1, 0, 0]

Sat($P_{>0.5}$ [G¬b]) = { s_1,s_3 }

Linear equation systems

- Solution of large (sparse) linear equation systems
  - size of system (number of variables) typically O(|S|)
  - state space S gets very large in practice
- Two main classes of solution methods:
  - direct methods – compute exact solutions in fixed number of steps, e.g. Gaussian elimination, L/U decomposition
  - iterative methods, e.g. Power, Jacobi, Gauss-Seidel, …
  - the latter are preferred in practice due to scalability
- General form: $\mathrm{A}\cdot\underline{x} = \underline{b}$
  - indexed over integers,
  - i.e. assume S = { 0,1,…,|S|-1 }

$$\sum_{j=0}^{|S|-1} \mathrm{A}(i,j)\cdot\underline{x}(j) = \underline{b}(i)$$

PCTL model checking - Summary

- Computation of set Sat(Φ) for DTMC D and PCTL formula Φ
  - recursive descent of parse tree
  - combination of graph algorithms, numerical computation
- Probabilistic operator P:
  - X Φ : one matrix-vector multiplication, $O(|S|^2)$
  - $\Phi_1\ U^{\leq k}\ \Phi_2$: k matrix-vector multiplications, $O(k|S|^2)$
  - $\Phi_1\ U\ \Phi_2$: linear equation system, at most |S| variables, $O(|S|^3)$
- Complexity:
  - linear in |Φ| and polynomial in |S|

Limitations of PCTL

- PCTL, although useful in practice, has limited expressivity
  - essentially: probability of reaching states in X, passing only through states in Y (and within k time-steps)

- More expressive logics can be used, for example:
  - LTL [Pnu77] – (non-probabilistic) linear-time temporal logic
  - PCTL* [ASB+95,BdA95] – which subsumes both PCTL and LTL
  - both allow path operators to be combined
  - (in PCTL, $P_{\sim p}$ [...] always contains a single temporal operator)
  - supported by PRISM
  - (not covered in this lecture)

- Another direction: extend DTMCs with costs and rewards…

Costs and rewards

- We augment DTMCs with rewards (or, conversely, costs)
  - real-valued quantities assigned to states and/or transitions
  - these can have a wide range of possible interpretations

- Some examples:
  - elapsed time, power consumption, size of message queue, number of messages successfully delivered, net profit, …

- Costs? or rewards?
  - mathematically, no distinction between rewards and costs
  - when interpreted, we assume that it is desirable to minimise costs and to maximise rewards
  - we will consistently use the terminology “rewards” regardless

Reward-based properties

- Properties of DTMCs augmented with rewards
  - allow a wide range of quantitative measures of the system
  - basic notion: expected value of rewards
  - formal property specifications will be in an extension of PCTL

- More precisely, we use two distinct classes of property…

- Instantaneous properties
  - the expected value of the reward at some time point

- Cumulative properties
  - the expected cumulated reward over some period

DTMC reward structures

- For a DTMC (S,s_{init},P,L), a reward structure is a pair $(\underline{\rho},\iota)$
  - $\underline{\rho} : S \rightarrow \mathbb{R}_{\geq 0}$ is the state reward function (vector)
  - $\iota : S \times S \rightarrow \mathbb{R}_{\geq 0}$ is the transition reward function (matrix)

- Example (for use with instantaneous properties)
  - "size of message queue": $\underline{\rho}$ maps each state to the number of jobs in the queue in that state, ι is not used

- Examples (for use with cumulative properties)
  - "time-steps": $\underline{\rho}$ returns 1 for all states and ι is zero (equivalently, $\underline{\rho}$ is zero and ι returns 1 for all transitions)
  - "number of messages lost": $\underline{\rho}$ is zero and ι maps transitions corresponding to a message loss to 1
  - "power consumption": $\underline{\rho}$ is defined as the per-time-step energy consumption in each state and ι as the energy cost of each transition

PCTL and rewards

- Extend PCTL to incorporate reward-based properties
  - add an R operator, which is similar to the existing P operator

expected reward is ~r

  - ϕ ::= … | $P_{\sim p}$ [ψ] | $R_{\sim r}$ [$I^{=k}$] | $R_{\sim r}$ [$C^{\leq k}$] | $R_{\sim r}$ [F ϕ]

“instantaneous” “cumulative” “reachability”

  - where $r \in \mathbb{R}_{\geq 0}$, $\sim \in \{<,>,\leq,\geq\}$, $k \in \mathbb{N}$

- $R_{\sim r}$ [·] means “the expected value of · satisfies ~r”

Types of reward formulas

- Instantaneous: $R_{\sim r}$ [$I^{=k}$]
  - “the expected value of the state reward at time-step k is ~r”
  - e.g. “the expected queue size after exactly 90 seconds”
- Cumulative: $R_{\sim r}$ [$C^{\leq k}$]
  - “the expected reward cumulated up to time-step k is ~r”
  - e.g. “the expected power consumption over one hour”
- Reachability: $R_{\sim r}$ [F ϕ]
  - “the expected reward cumulated before reaching a state satisfying ϕ is ~r”
  - e.g. “the expected time for the algorithm to terminate”

Reward formula semantics

- Formal semantics of the three reward operators
  - based on random variables over (infinite) paths

- Recall:
  - $s \vDash P_{\sim p} [\ \psi\ ] \ \Leftrightarrow\ Pr_s \{\ \omega \in Path(s) \mid \omega \vDash \psi\ \} \sim p$

- For a state s in the DTMC (see [KNP07a] for full definition):
  - $s \vDash R_{\sim r} [\ I^{=k}\ ] \ \Leftrightarrow\ Exp(s, X_{I=k}) \sim r$
  - $s \vDash R_{\sim r} [\ C^{\leq k}\ ] \ \Leftrightarrow\ Exp(s, X_{C\leq k}) \sim r$
  - $s \vDash R_{\sim r} [\ F\ \Phi\ ] \ \Leftrightarrow\ Exp(s, X_{F\Phi}) \sim r$

  where: Exp(s, X) denotes the expectation of the random variable $X : Path(s) \rightarrow \mathbb{R}_{\geq 0}$ with respect to the probability measure Pr_s

Reward formula semantics

- Definition of random variables:
  - for an infinite path $\omega = s_0 s_1 s_2 \ldots$

$$X_{I=k}(\omega) \;=\; \underline{\rho}(s_k)$$

$$X_{C\leq k}(\omega) \;=\begin{cases} 0 & \text{if } k=0 \\ \sum_{i=0}^{k-1} \underline{\rho}(s_i) + \iota(s_i, s_{i+1}) & \text{otherwise} \end{cases}$$

$$X_{F\phi}(\omega) \;=\begin{cases} 0 & \text{if } s_0 \in Sat(\phi) \\ \infty & \text{if } s_i \notin Sat(\phi) \text{ for all } i \geq 0 \\ \sum_{i=0}^{k_\phi-1} \underline{\rho}(s_i) + \iota(s_i, s_{i+1}) & \text{otherwise} \end{cases}$$

  - where $k_\phi = \min\{\, j \mid s_j \vDash \phi \,\}$

Model checking reward properties

- Instantaneous: $R_{\sim r}$ [$I^{=k}$]
- Cumulative: $R_{\sim r}$ [$C^{\leq k}$]
  - variant of the method for computing bounded until probabilities
  - solution of recursive equations
- Reachability: $R_{\sim r}$ [F ϕ]
  - similar to computing until probabilities
  - precomputation phase (identify infinite reward states)
  - then reduces to solving a system of linear equation
- For more details, see e.g. [KNP07a]
  - complexity not increased wrt classical PCTL

Overview (Part 1)

- Probability basics
- Model checking for discrete-time Markov chains (DTMCs)
  - DTMCs: definition, paths & probability spaces
  - PCTL model checking
  - Costs and rewards
- A glimpse of model checking for continuous-time Markov chains (CTMCs)
- PRISM: overview
  - Modelling language
  - Properties
  - GUI, etc
  - Case study: Bluetooth device discovery
- Summary

Continuous-time Markov chains

- Continuous-time Markov chains (CTMCs)
  - labelled transition systems augmented with rates
  - discrete states, continuous time-steps
  - delays exponentially distributed

- Formally, a CTMC C is a tuple (S, s_{init}, R, L) where:
  - S is a finite set of states ("state space")
  - $s_{init} \in S$ is the initial state
  - $R : S \times S \rightarrow \mathbb{R}_{\geq 0}$ is the transition rate matrix
  - $L : S \rightarrow 2^{AP}$ is a labelling with atomic propositions

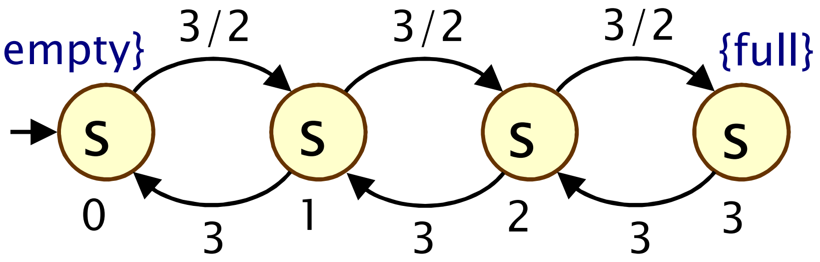
- Transition rates
  - transition between s and s' when $R(s,s')>0$
  - probability triggered before t time units $1 - e^{-R(s,s')\cdot t}$

Simple CTMC example

$C = (S, s_{init}, R, L)$

$S = \{s_0, s_1, s_2, s_3\}$

$s_{init} = s_0$



$AP = \{empty, full\}$

$L(s_0)=\{empty\}$ $L(s_1)=L(s_2)=\varnothing$ and $L(s_3)=\{full\}$

$$R = \begin{bmatrix} 0 & 3/2 & 0 & 0 \\ 3 & 0 & 3/2 & 0 \\ 0 & 3 & 0 & 3/2 \\ 0 & 0 & 3 & 0 \end{bmatrix} \qquad Q = \begin{bmatrix} -3/2 & 3/2 & 0 & 0 \\ 3 & -9/2 & 3/2 & 0 \\ 0 & 3 & -9/2 & 3/2 \\ 0 & 0 & 3 & -3 \end{bmatrix}$$

transition rate matrix

infinitesimal generator matrix

Transient and steady-state behaviour

- Transient behaviour
  - state of the model at a particular time instant
  - $\underline{\pi}_{s,t}(s')$ is probability of, having started in state s, being in state s' at time t
  - $\underline{\pi}_{s,t}(s') = Pr_s\{\ \omega \in Path(s) \mid \omega@t=s'\ \}$
- Steady-state behaviour
  - state of the model in the long-run
  - $\underline{\pi}_s(s')$ is probability of, having started in state s, being in state s' in the long run
  - $\underline{\pi}_s(s') = \lim_{t\to\infty} \underline{\pi}_{s,t}(s')$
  - the percentage of time, in long run, spent in each state
- Can compute these numerically, from rates matrix R
  - e.g. embedded/uniformised DTMC

Temporal logic CSL

- CSL – Continuous Stochastic Logic
- Similar to PCTL, except real-valued time
  - $P_{=?}$ [$F^{[4,5.6]}$ outOfPower] – the (transient) probability of being out of power in time interval of 4 to 5.6 time units
  - $S_{=?}$ [minQoS] – the steady-state probability of satisfying minimum QoS
  - $R_{<10}$ [$C^{\leq 5}$] – cumulated reward up to time 5 is less than 10
- Model checking proceeds essentially via discretisation…
  - discretise CTMC to obtain DTMC (embedded, uniformised)
  - combine with graph-theoretical analysis
- State-space explosion
  - can we exploit continuous approximations?

Overview (Part 1)

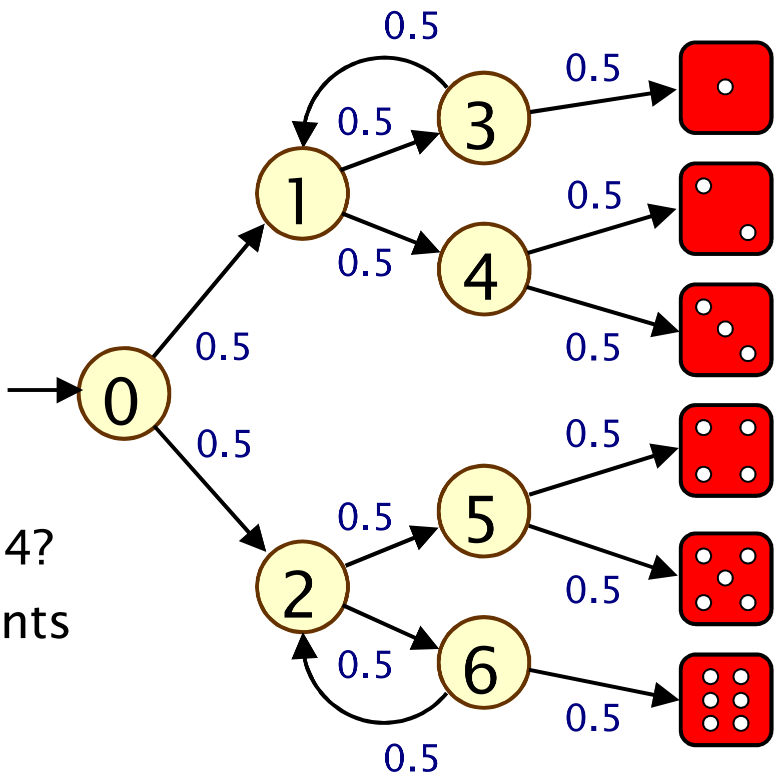
- Probability basics
- Model checking for discrete-time Markov chains (DTMCs)
  - DTMCs: definition, paths & probability spaces
  - PCTL model checking
  - Costs and rewards
- A glimpse of model checking for continuous-time Markov chains (CTMCs)
- PRISM: overview
  - Modelling language
  - Properties
  - GUI, etc
  - Case study: Bluetooth device discovery
- Summary

PRISM…

- Model checking for various temporal logics…
  - probabilistic/reward extensions of CTL/CTL*/LTL
  - PCTL, CSL, LTL, PCTL*, rPATL, CTL, …
- Various efficient model checking engines and techniques
  - symbolic methods (binary decision diagrams and extensions)
  - explicit-state methods (sparse matrices, etc.)
  - statistical model checking (simulation-based approximations)
  - and more: symmetry reduction, quantitative abstraction refinement, fast adaptive uniformisation, ...
- Graphical user interface
  - editors, simulator, experiments, graph plotting
- See: http://www.prismmodelchecker.org/
  - downloads, tutorials, case studies, papers, …

Probability example

- Modelling a 6-sided die using a fair coin
  - algorithm due to Knuth/Yao:
  - start at 0, toss a coin
  - upper branch when H
  - lower branch when T
  - repeat until value chosen

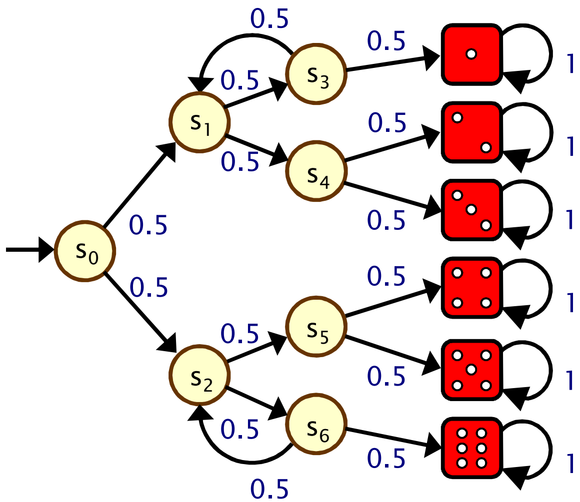


- Is this algorithm correct?
  - e.g. probability of obtaining a 4?
  - obtain as disjoint union of events
  - THH, TTTHH, TTTTTHH, …
  - Pr(“eventually 4”)

    $= (1/2)^3 + (1/2)^5 + (1/2)^7 + \ldots = 1/6$

Example…

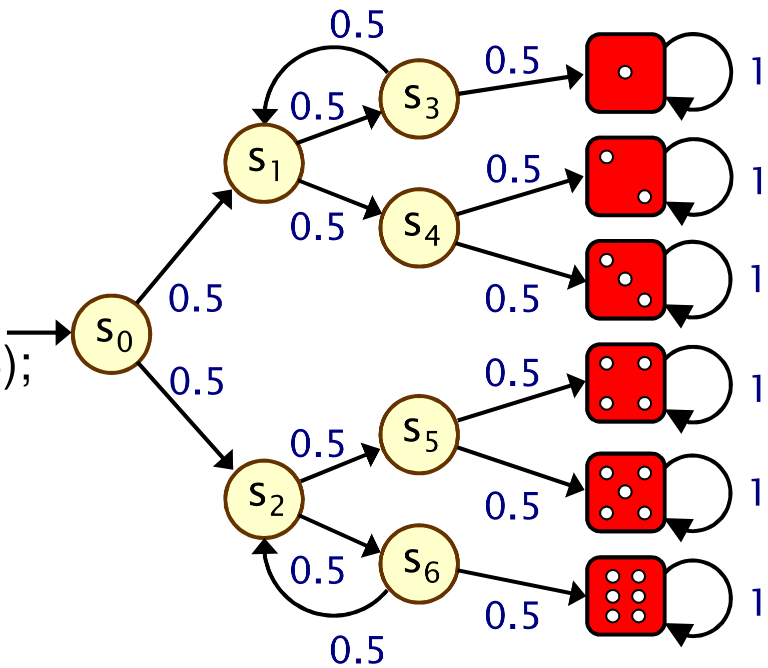
- Other properties?
  - "what is the probability of termination?"
- e.g. efficiency?
  - "what is the probability of needing more than 4 coin tosses?"
  - "on average, how many coin tosses are needed?"



- Probabilistic model checking provides a framework for these kinds of properties…
  - modelling languages
  - property specification languages
  - model checking algorithms, techniques and tools

Probabilistic models

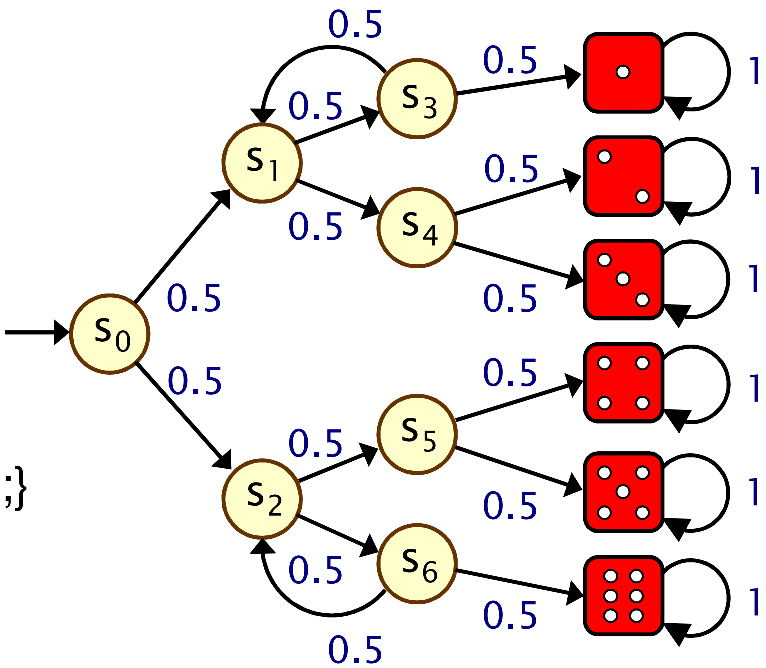
```
dtmc
module die
// local state s : [0..7] init 0;
// value of the dice d : [0..6] init 0;
[] s=0 -> 0.5 : (s'=1) + 0.5 : (s'=2);
…
[] s=3 ->
   0.5 : (s'=1) + 0.5 : (s'=7) & (d'=1);
[] s=4 ->
   0.5 : (s'=7) & (d'=2) + 0.5 : (s'=7) & (d'=3);
…
[] s=7 -> (s'=7);
endmodule
rewards "coin_flips"
[] s<7 : 1;
endrewards
```



Given in PRISM’s guarded commands modelling notation

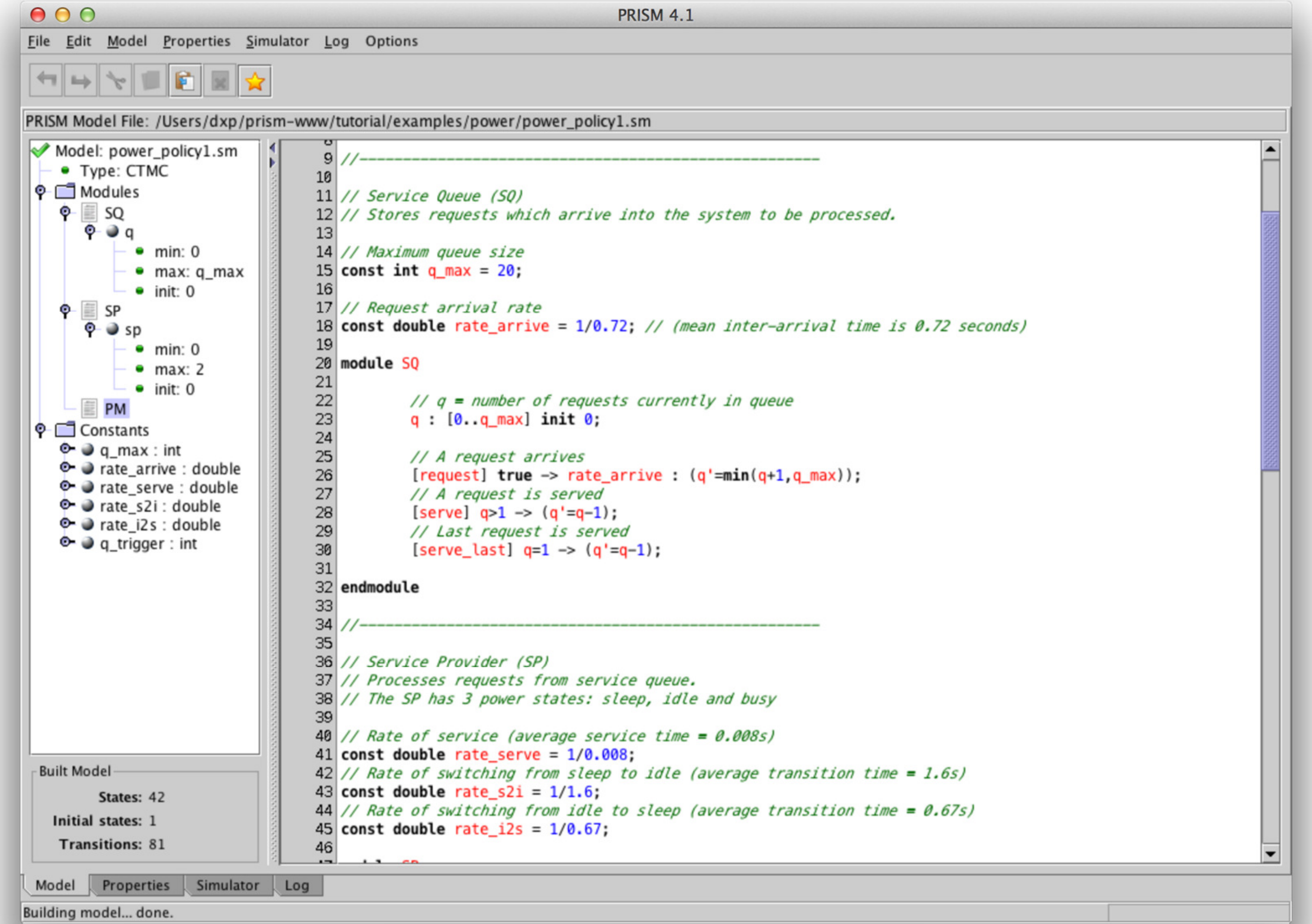
Probabilistic models

```
int s, d;
s = 0; d = 0;
while (s < 7) {
  bool coin = Bernoulli(0.5);
  if (s = 0)
    if (coin) s = 1 else s = 2;
...
  else if (s = 3)
    if (coin) s = 1 else {s = 7; d = 1;}
  else if (s = 4)
    if (coin) {s = 7; d = 2} else {s = 7; d = 3;}
…
}
return (d)
```

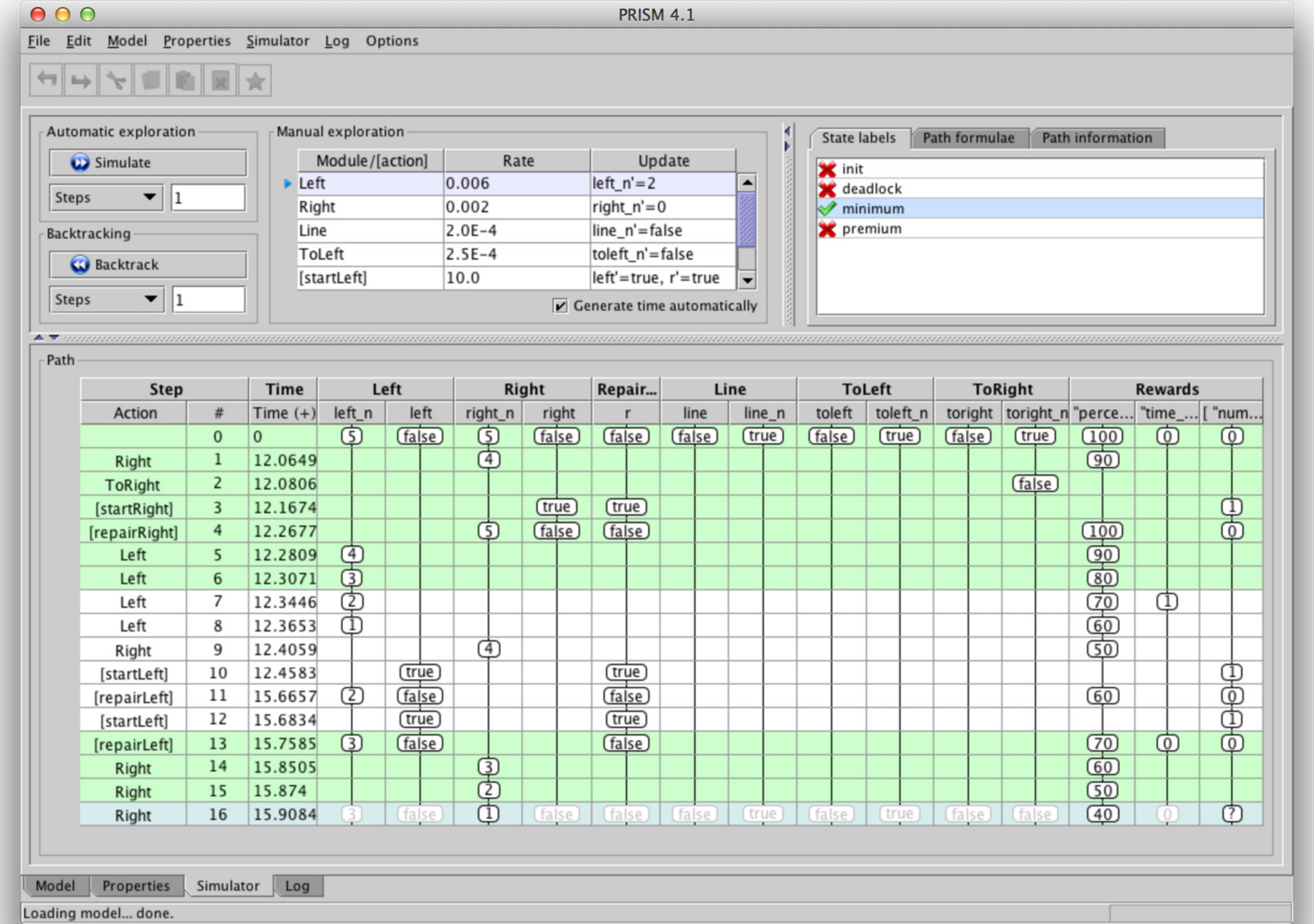


Given as a probabilistic program

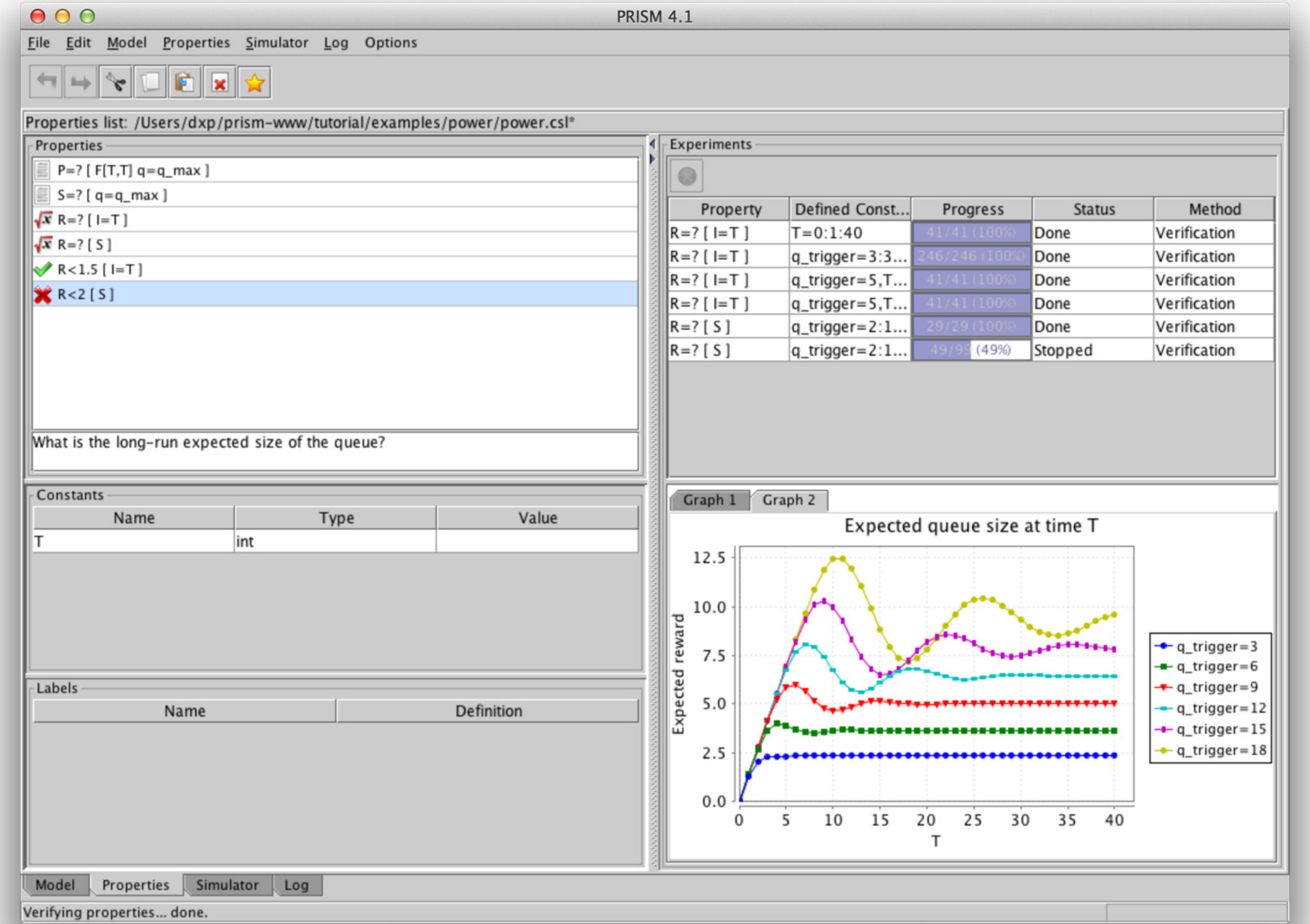
PRISM GUI: Editing a model



PRISM GUI: The Simulator



PRISM GUI: Model checking and graphs



PRISM – Case studies

- Randomised distributed algorithms
  - consensus, leader election, self-stabilisation, …
- Randomised communication protocols
  - Bluetooth, FireWire, Zeroconf, 802.11, Zigbee, gossiping, …
- Security protocols/systems
  - contract signing, anonymity, pin cracking, quantum crypto, …
- Biological systems
  - cell signalling pathways, DNA computation, …
- Planning & controller synthesis
  - robotics, dynamic power management, …
- Performance & reliability
  - nanotechnology, cloud computing, manufacturing systems, …

- See: www.prismmodelchecker.org/casestudies

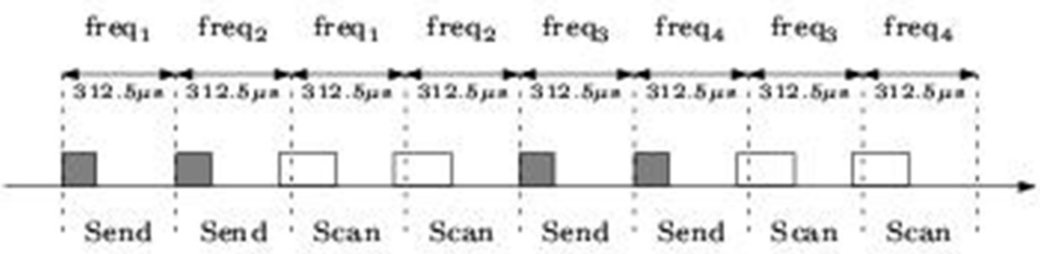
Case study: Bluetooth device discovery

- Bluetooth: short-range low-power wireless protocol
  - widely available in phones, PDAs, laptops, ...
  - open standard, specification freely available
- Uses frequency hopping scheme
  - to avoid interference (uses unregulated 2.4GHz band)
  - pseudo-random selection over 32 of 79 frequencies
- Formation of personal area networks (PANs)
  - piconets (1 master, up to 7 slaves)
  - self-configuring: devices discover themselves
- Device discovery
  - mandatory first step before any communication possible
  - relatively high power consumption so performance is crucial
  - master looks for devices, slaves listens for master

Master (sender) behaviour

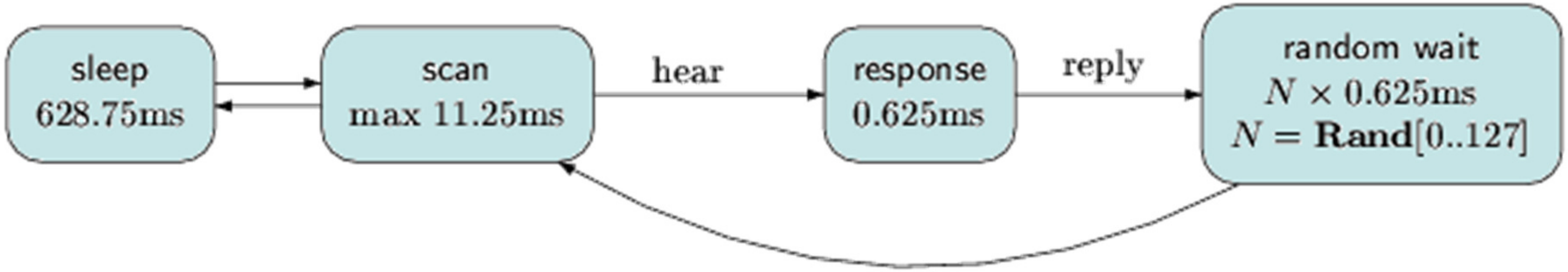
- 28 bit free-running clock CLK, ticks every 312.5µs
- Frequency hopping sequence determined by clock:
  - freq = $[CLK_{16-12}+k+(CLK_{4-2,0}-CLK_{16-12}) \bmod 16] \bmod 32$
  - 2 trains of 16 frequencies (determined by offset k), 128 times each, swap between every 2.56s
- Broadcasts “inquiry packets” on two consecutive frequencies, then listens on the same two

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
17	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	2	19	20	21	22	23	24	25	26	27	28	29	30	31	32
1	2	3	20	21	22	23	24	25	26	27	28	29	30	31	32
17	18	19	20	5	6	7	8	9	10	11	12	13	14	15	16
17	18	19	20	21	6	7	8	9	10	11	12	13	14	15	16
1	2	3	4	5	6	23	24	25	26	27	28	29	30	31	32
1	2	3	4	5	6	7	24	25	26	27	28	29	30	31	32
17	18	19	20	21	22	23	24	9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24	25	10	11	12	13	14	15	16
1	2	3	4	5	6	7	8	9	10	27	28	29	30	31	32
1	2	3	4	5	6	7	8	9	10	11	28	29	30	31	32
17	18	19	20	21	22	23	24	25	26	27	28	13	14	15	16
17	18	19	20	21	22	23	24	25	26	27	28	29	14	15	16
1	2	3	4	5	6	7	8	9	10	11	12	13	14	31	32
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	32
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
1	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
17	18	3	4	5	6	7	8	9	10	11	12	13	14	15	16
17	18	19	4	5	6	7	8	9	10	11	12	13	14	15	16
1	2	3	4	21	22	23	24	25	26	27	28	29	30	31	32
1	2	3	4	5	22	23	24	25	26	27	28	29	30	31	32
17	18	19	20	21	22	7	8	9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	8	9	10	11	12	13	14	15	16
1	2	3	4	5	6	7	8	25	26	27	28	29	30	31	32
1	2	3	4	5	6	7	8	9	26	27	28	29	30	31	32
17	18	19	20	21	22	23	24	25	26	11	12	13	14	15	16
17	18	19	20	21	22	23	24	25	26	27	12	13	14	15	16
1	2	3	4	5	6	7	8	9	10	11	12	29	30	31	32
1	2	3	4	5	6	7	8	9	10	11	12	13	30	31	32
17	18	19	20	21	22	23	24	25	26	27	28	29	30	15	16
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	16



Slave (receiver) behaviour

- Listens (scans) on frequencies for inquiry packets
  - must listen on right frequency at right time
  - cycles through frequency sequence at much slower speed (every 1.28s)



- On hearing packet, pause, send reply and then wait for a random delay before listening for subsequent packets
  - avoid repeated collisions with other slaves

Bluetooth – PRISM model

- Modelled/analysed using PRISM model checker [DKNP06]
  - model scenario with one sender and one receiver
  - synchronous (clock speed defined by Bluetooth spec)
  - model at lowest-level (one clock-tick = one transition)
  - randomised behaviour so model as a DTMC
  - use real values for delays, etc. from Bluetooth spec

- Modelling challenges
  - complex interaction between sender/receiver
  - combination of short/long time-scales – cannot scale down
  - sender/receiver not initially synchronised, so huge number of possible initial configurations (17,179,869,184)

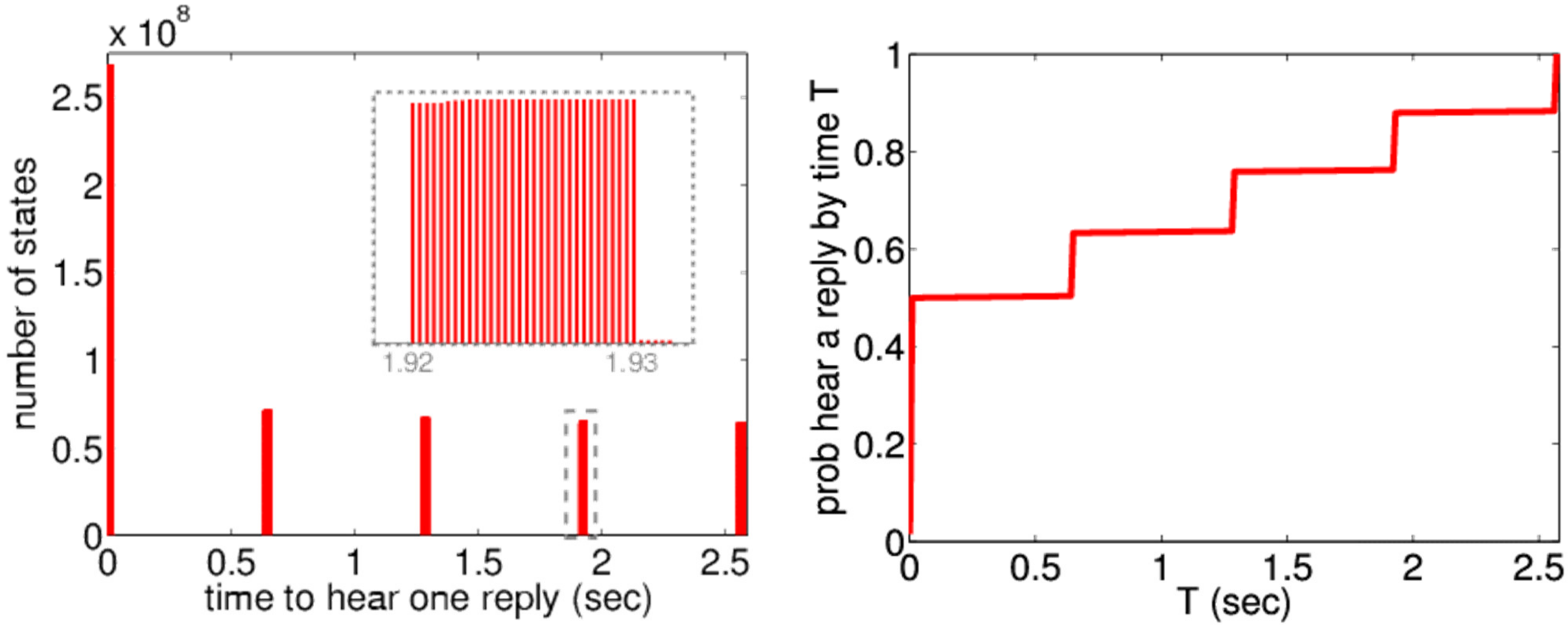
Bluetooth – Results

- Huge DTMC – initially, model checking infeasible
  - partition into 32 scenarios, i.e. 32 separate DTMCs
  - on average, approx. 3.4×10^9 states (536,870,912 initial)
  - can be built/analysed with PRISM's MTBDD engine

- We compute:
  - R=? [F replies=K {"init"}{max}]
  - "worst-case expected time to hear K replies over all possible initial configurations"

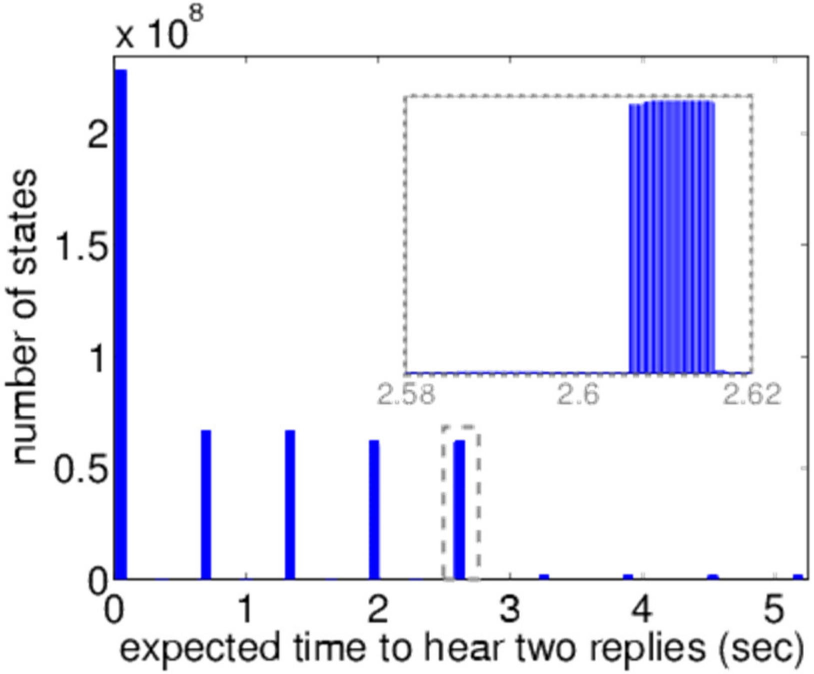
- Also look at:
  - how many initial states for each possible expected time
  - cumulative distribution function (CDF) for time, assuming equal probability for each initial state

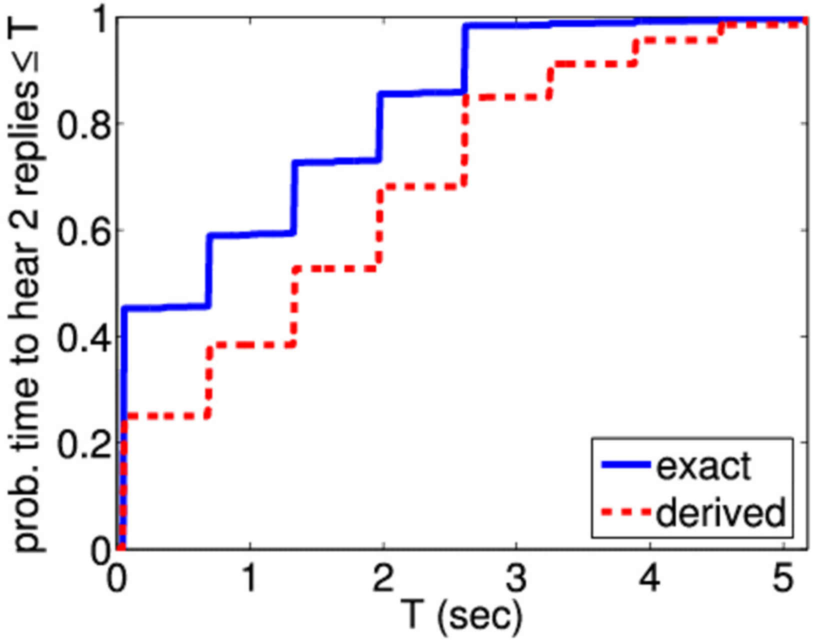
Bluetooth – Time to hear 1 reply



- Worst-case expected time = 2.5716 sec
  - in 921,600 possible initial states
  - best-case = 635 μs

Bluetooth – Time to hear 2 replies





- Worst-case expected time = 5.177 sec
  - in 444 possible initial states
  - compare actual CDF with derived version which assumes times to reply to first/second messages are independent

Summary (Part 1)

- Discrete-time Markov chains (DTMCs)
  - state transition systems + discrete probabilistic choice
  - probability space over paths through a DTMC
- Property specifications
  - probabilistic extensions of temporal logic, e.g. PCTL, LTL
  - also: expected value of costs/rewards
- Model checking algorithms
  - combination of graph-based algorithms, numerical computation, automata constructions
  - also applicable to continuous-time Markov chains via discretisation
- PRISM and Bluetooth case study

- Next: Markov decision processes (MDPs)